An algorithm for computing the canonical bases of higher-level q-deformed Fock spaces

Xavier YVONNE

Abstract

We derive a straightening-free algorithm that computes the canonical bases of any higher-level q-deformed Fock space.

Keywords: Quantum groups, canonical bases, Fock spaces.

1 Introduction

The higher-level q-deformed Fock spaces form an important family of integrable representations of the quantum group $U_q(\widehat{\mathfrak{sl}}_n)$. The Fock representation $\mathbf{F}_q[\boldsymbol{s}_l]$ depends on a parameter $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ called multi-charge. It was introduced in [JMMO] in order to compute the crystal graph of the integrable irreducible representation of $U_q(\widehat{\mathfrak{sl}}_n)$ with highest weight $\Lambda_{s_1} + \cdots + \Lambda_{s_l}$.

The canonical basis of a Fock space is a lower global crystal basis in the sense of [Kas1] that enjoys nice properties. It was constructed for $l = 1$ in [LT1, LT2] and for $l \geq 1$ by Uglov [U]. This construction relies on

1. an embedding of $U_q(\widehat{\mathfrak{sl}}_n)$-modules: $\mathbf{F}_q[\boldsymbol{s}_l] \hookrightarrow \Lambda^s$ which is a bijection if (and only if) $l = 1$. Here Λ^s is the space of semi-infinite q-deformed wedge products of charge $s = s_1 + \cdots + s_l$ (see [U]).

2. the definition of a $^-$-involution of Λ^s compatible with the embedding above. The canonical basis is then the unique $^-$-invariant basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ satisfying a certain congruence property modulo the $\mathbb{Z}[q]$-lattice spanned by l-multi-partitions.

In [U], Uglov provides an algorithm for computing the canonical basis based on the straightening of non-ordered q-wedge products. Unfortunately, in particular when $s_1 \gg \cdots \gg s_l$, the number of q-wedge products (of 2 factors) to be straightened becomes too large for this algorithm being used for practical computations.

The goal of this paper is to derive a faster algorithm, which does not require straightening q-wedge products. First, we compute a $^-$-invariant basis of Λ^s. The vectors of this basis are obtained by letting act the generators of the quantum groups $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ (with $p = -q^{-1}$) and a Heisenberg algebra $\mathcal{H}$ to the vacuum vector (see Thm. 3.11). Then, we compute the intersection of this basis with a given weight subspace V of $\mathbf{F}_q[\boldsymbol{s}_l]$ (see Prop. 3.16). We thus get the transition matrix $T(q)$ between the standard basis and a $^-$-invariant

basis of V. We can then compute the matrix $A(q)$ of the involution of V with respect to the standard basis by the formula $A(q) = T(q)\big(T(q^{-1})\big)^{-1}$. Note that we are forced here to invert a large matrix with Laurent polynomial entries. This is the time-consuming step of our algorithm. Note also that although $A(q)$ is unitriangular by [U], this matrix $T(q^{-1})$ is not. Once $A(q)$ is known, the computation of the canonical basis is pure routine (see *e.g.* [L, Thm. 7.1]).

Our algorithm is a generalization of the algorithm of Leclerc and Thibon (see [L]) for $l = 1$. However, two difficulties arise when $l > 1$. First, we have to take into account the action of $U_p(\widehat{\mathfrak{sl}}_l)$, which is trivial if $l = 1$. Secondly, the action of the Heisenberg algebra $\mathcal{H}$ is much more complicated for $l > 1$ and it particular it requires some new work to calculate its action on the vacuum vectors without using the straightening relations (see Section 3.3).

Using our algorithm, we were able to compute the canonical bases of large weight subspaces of different Fock spaces (see examples in Section 4). These calculations helped us to conjecture and prove a theorem giving a combinatorial expression of the derivative at $q = 1$ of the $^-$-involution (see [Y1, Thm. 2.11]). The latter result supports in turn a new conjecture for computing the q-decomposition matrices of Dipper-James-Mathas' cyclotomic v-Schur algebras with parameters specialized at powers of a complex n-th root of unity ([DJM, Y1]). In this context, our result is an analogue of the Jantzen sum formula for cyclotomic v-Schur algebras [JM].

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Notation. Let $\mathbb{N}$ (*resp.* $\mathbb{N}^*$) denote the set of nonnegative (*resp.* positive) integers, and for $a,\, b \in \mathbb{R}$ denote by $[\![a;b]\!]$ the discrete interval $[a;b] \cap \mathbb{Z}$. For $X \subset \mathbb{R}$, $a \in \mathbb{R}$, $N \in \mathbb{N}^*$, put

$$X^N(a) := \{(x_1, \dots, x_N) \in X^N \mid x_1 + \cdots + x_N = a\}. \tag{1}$$

Throughout this article, we fix 3 integers $n,\, l \geq 1$ and $s \in \mathbb{Z}$. Let Π denote the set of all integer partitions. If (W, S) is a Coxeter system, denote by $\ell : W \to \mathbb{N}$ the length function on W.

2 Higher-level q-deformed Fock spaces

In this section, we recall briefly the definition of the higher-level Fock spaces and their canonical bases. These objects were introduced by [U], to which we refer the reader for more details. We follow here the notation from [Y2].

2.1 The quantum algebras $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$

In this section, we assume that $n \geq 2$ and $l \geq 2$. Let $\widehat{\mathfrak{sl}}_n$ be the Kac-Moody algebra of type $A^{(1)}_{n-1}$ defined over the field $\mathbb{Q}$ [Kac]. Let $\Lambda_0, \dots, \Lambda_{n-1}$ be the fundamental weights, $\alpha_0, \dots, \alpha_{n-1}$ be the simple roots and $\delta := \alpha_0 + \cdots + \alpha_{n-1}$ be the null root. It will be

convenient to extend the index set of the fundamental weights by setting $\Lambda_i := \Lambda_{i \bmod n}$ for all $i \in \mathbb{Z}$. The space $\bigoplus_{i=0}^{n-1} \mathbb{Q}\,\Lambda_i \;\oplus\; \mathbb{Q}\,\delta = \bigoplus_{i=0}^{n-1} \mathbb{Q}\,\alpha_i \;\oplus\; \mathbb{Q}\,\Lambda_0$ is equipped with a non-degenerate bilinear symmetric form $(.\,,\,.)$ defined by

$$(\alpha_i, \alpha_j) = a_{i,j}, \quad (\Lambda_0, \alpha_i) = \delta_{i,0}, \quad (\Lambda_0, \Lambda_0) = 0 \qquad (0 \le i, j \le n-1), \tag{2}$$

where $(a_{i,j})_{0 \le i,j \le n-1}$ is the Cartan matrix of $\widehat{\mathfrak{sl}}_n$. Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the q-deformed universal enveloping algebra of $\widehat{\mathfrak{sl}}_n$ (see *e.g.* [KMS, U]). This is an algebra over $\mathbb{Q}(q)$ with generators e_i, f_i, $t_i^{\pm 1}$ $(0 \le i \le n-1)$ and ∂; the relations will be omitted. The subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by e_i, f_i, $t_i^{\pm 1}$ $(0 \le i \le n-1)$ will be denoted by $U_q'(\widehat{\mathfrak{sl}}_n)$. If M is a $U_q(\widehat{\mathfrak{sl}}_n)$-module, denote by $\mathcal{P}(M)$ the set of weights of M and let $M\langle w\rangle$ denote the subspace of M of weight w. Let

$$\mathrm{wt}(x) := w \tag{3}$$

denote the weight of $x \in M\langle w\rangle$. The Weyl group of $\widehat{\mathfrak{sl}}_n$ (or $U_q(\widehat{\mathfrak{sl}}_n)$) is

$$W_n = \langle \sigma_0, \dots, \sigma_{n-1}\rangle \cong \widetilde{\mathfrak{S}}_n, \tag{4}$$

where σ_i $(0 \le i \le n-1)$ is the orthogonal reflection that fixes pointwise the hyperplane orthogonal to α_i.

We also introduce the algebra $U_p(\widehat{\mathfrak{sl}}_l)$ with

$$p := -q^{-1}. \tag{5}$$

In order to distinguish the elements related to $U_q(\widehat{\mathfrak{sl}}_n)$ from those related to $U_p(\widehat{\mathfrak{sl}}_l)$, we put dots over the latter. For example, $\dot{e}_i$, $\dot{f}_i$, $\dot{t}_i^{\pm 1}$ $(0 \le i \le l-1)$ and $\dot{\partial}$ are the generators, $\dot{\alpha}_i$ $(0 \le i \le l-1)$ are the simple roots, $\dot{W}_l = \langle \dot{\sigma}_0, \dots, \dot{\sigma}_{l-1}\rangle$ is the Weyl group of $U_p(\widehat{\mathfrak{sl}}_l)$ and so on.

2.2 The space Λ^s

Following [U], let

$$\Lambda^s = \Lambda^s[n, l] \tag{6}$$

be the space of (semi-infinite) q-wedge products of charge s (this space is denoted by $\Lambda^{s+\frac{\infty}{2}}$ in [U]). This vector space has a natural basis formed by the ordered q-wedge products; this basis is called *standard*. A non-ordered q-wedge product can be expressed as a linear combination of ordered q-wedge products by using the straightening relations given in [U, Prop. 3.16] (we do not need these relations in this article). Recall the notation $\mathbb{Z}^l(s)$ and $\mathbb{Z}^n(s)$ from (1). Following [U, §4.1] or [Y2, §2.2.1], we shall use in the sequel the following indexations of the standard basis:

$$\{|\lambda, s\rangle \mid \lambda \in \Pi\} = \{|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l,\ \boldsymbol{s}_l \in \mathbb{Z}^l(s)\} = \{|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet} \mid \boldsymbol{\lambda}_n \in \Pi^n,\ \boldsymbol{s}_n \in \mathbb{Z}^n(s)\}. \tag{7}$$

Following [JMMO, FLOTW, U], the vector space Λ^s can be made into an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$. The action of $U_q(\widehat{\mathfrak{sl}}_n)$ can be described in a nice way in terms of addable/removable i-nodes of l-multi-partitions; see [U, Eq. (33-34)]. Note that these formulas do no involve the straightening of q-wedge products; they are therefore handy to use for computations. In a completely similar way, Λ^s can be made into an integrable representation of level n of the quantum algebra $U_p(\widehat{\mathfrak{sl}}_l)$. This action can be described using the indexation by n-multi-partitions; see [U, Eq. (35-36)]. The vectors of the standard basis of Λ^s are weight vectors for the actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$. In order to recall the expression of the weights, let us introduce the following notation. For $\boldsymbol{\lambda}_l \in \Pi^l$, $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $0 \le i \le n-1$, denote by $N_i(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)$ the number of nodes with residue i (modulo n) that are contained in the Young diagram of $\boldsymbol{\lambda}_l$ (note that the definition of residues involves the multi-charge $\boldsymbol{s}_l$). For $\boldsymbol{s}_l = (s_1, \ldots, s_l) \in \mathbb{Z}^l$, define

$$\Delta(\boldsymbol{s}_l, n) := \frac{1}{2}\sum_{b=1}^{l}\Big(\frac{s_b^2}{n} - s_b\Big) - \Big(\frac{(s_b \bmod n)^2}{n} - (s_b \bmod n)\Big), \tag{8}$$

where for $1 \le b \le n$, $s_b \bmod n$ denotes the integer in $[\![0; n-1]\!]$ that is congruent to s_b modulo n.

Proposition 2.1 ([U]) *With the notation above, we have*

$$\mathrm{wt}(|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle) = -\Delta(\boldsymbol{s}_l, n)\delta + \Lambda_{s_1} + \cdots + \Lambda_{s_l} - \sum_{i=0}^{n-1} N_i(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)\,\alpha_i, \tag{9}$$

$$\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle) = -\big(\Delta(\boldsymbol{s}_l, n) + N_0(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)\big)\dot{\delta} + (n - s_1 + s_l)\dot{\Lambda}_0 + \sum_{i=1}^{l-1}(s_i - s_{i+1})\,\dot{\Lambda}_i, \tag{10}$$

$$\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}) = -\Delta(\boldsymbol{s}_n, l)\dot{\delta} + \dot{\Lambda}_{s_1} + \cdots + \dot{\Lambda}_{s_n} - \sum_{i=0}^{l-1} N_i(\boldsymbol{\lambda}_n; \boldsymbol{s}_n; l)\,\dot{\alpha}_i, \tag{11}$$

$$\mathrm{wt}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}) = -\big(\Delta(\boldsymbol{s}_n, l) + N_0(\boldsymbol{\lambda}_n; \boldsymbol{s}_n; l)\big)\delta + (l - s_1 + s_n)\Lambda_0 + \sum_{i=1}^{n-1}(s_i - s_{i+1})\,\Lambda_i. \tag{12}$$

$\square$

For $m \in \mathbb{Z}^*$, Uglov defined an endomorphism B_m of Λ^s (see [U, Eq. (25) & Sect. 4.3] or [Y2, Def. 2.2]). His definition is obtained by taking the limit $r \to \infty$ in the action of the center of the Hecke algebra of $\widehat{\mathfrak{S}}_r$ on q-wedge products of r factors. However, by [U] the operators B_m do not commute, but they span a Heisenberg algebra

$$\mathcal{H} = \langle B_m \mid m \in \mathbb{Z}^* \rangle. \tag{13}$$

Note that the q-wedge products involved in the definition of B_m are in general not ordered. Therefore, the computation of the action of $\mathcal{H}$ often requires straightening many q-wedge products.

We now recall some results concerning the actions of $U_q(\widehat{\mathfrak{sl}}_n)$, $U_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ on Λ^s.

Proposition 2.2 ([U]) *The actions of* $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ *and* $\mathcal{H}$ *on* Λ^s *pairwise commute.* □

For L, $N \in \mathbb{N}^*$, introduce the finite set

$$A_{L,N}(s) := \{(r_1, \dots, r_L) \in \mathbb{Z}^L(s) \mid r_1 \geq \cdots \geq r_L,\ r_1 - r_L \leq N\}. \tag{14}$$

Theorem 2.3 ([U], Thm. 4.8) *We have*

$$\Lambda^s = \bigoplus_{\boldsymbol{r}_l \in A_{l,n}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = \bigoplus_{\boldsymbol{r}_n \in A_{n,l}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet.$$

□

2.3 Higher-level q-deformed Fock spaces

Recall that $p = -q^{-1}$. For $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$, let

$$\mathbf{F}_q[\boldsymbol{s}_l] = \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{Q}(q)|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle, \qquad \mathbf{F}_p[\boldsymbol{s}_n]^\bullet = \bigoplus_{\boldsymbol{\lambda}_n \in \Pi^n} \mathbb{Q}(q)|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet \tag{15}$$

denote the higher-level (q-deformed) Fock spaces [U]. By [U], the subspace $\mathbf{F}_q[\boldsymbol{s}_l] \subset \Lambda^s$ is stable under the actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{H}$ (but not under the action of $U_p(\widehat{\mathfrak{sl}}_l)$) and the subspace $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet \subset \Lambda^s$ is stable under the actions of $U_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$.

For N, $L \in \mathbb{N}^*$, define a bijective map

$$\theta_{L,N} : \mathbb{Q}^L(s) \to \mathbb{Q}^L(N), \quad (s_1, \dots, s_L) \mapsto (N - s_1 + s_L, s_1 - s_2, \dots, s_{L-1} - s_L). \tag{16}$$

The next result shows that the Fock spaces are sums of certain weight subspaces of Λ^s. The proof follows easily from Proposition 2.1.

Proposition 2.4 ([U])

(i) *Let* $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$. *Let* $(a_0, \dots, a_{n-1}) := \theta_{n,l}(\boldsymbol{s}_n)$ *and* $w := \sum_{i=0}^{n-1} a_i \Lambda_i$. *Then we have* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet = \bigoplus_{d \in \mathbb{Z}} \Lambda^s\langle w + d\delta\rangle$.

(ii) *Let* $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. *Let* $(a_0, \dots, a_{l-1}) := \theta_{l,n}(\boldsymbol{s}_l)$ *and* $\dot{w} := \sum_{i=0}^{l-1} a_i \dot{\Lambda}_i$. *Then we have* $\mathbf{F}_q[\boldsymbol{s}_l] = \bigoplus_{d \in \mathbb{Z}} \Lambda^s\langle \dot{w} + d\dot{\delta}\rangle$. □

We now compare some weight subspaces of the Fock spaces. The proof again follows from Proposition 2.1.

Proposition 2.5 *Let* $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ *and* w *be a weight of* $\mathbf{F}_q[\boldsymbol{s}_l]$. *Then there exists a unique pair* $(\boldsymbol{s}_n, \dot{w})$ *such that* $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$, $\dot{w}$ *is a weight of* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ *and* $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle = \mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle \dot{w}\rangle$. *More precisely, write* $w = d\delta + \sum_{i=0}^{n-1} a_i \Lambda_i$ *(with* $a_0, \dots, a_{n-1}, d \in \mathbb{Z}$*),* $\boldsymbol{s}_l = (s_1, \dots, s_l)$ *and put* $s_0 := n + s_l$. *Then we have* $\boldsymbol{s}_n = \theta_{n,l}^{-1}(a_0, \dots, a_{n-1})$ *and* $\dot{w} = d\dot{\delta} + \sum_{i=0}^{l-1} (s_i - s_{i+1})\dot{\Lambda}_i$. □

Example 2.6 Take $n=3$, $l=2$, $\boldsymbol{s}_l=(1,0)$ and $w=-2\Lambda_0+\Lambda_1+3\Lambda_2-2\delta$. Then by (9), we have $\mathrm{wt}\big(\big|\big((1,1),(1)\big),\boldsymbol{s}_l\big\rangle\big)=w$, so w is a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. By Proposition 2.5, we have $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot w\rangle$ with $\boldsymbol{s}_n=(2,1,-2)$ and $\dot w=2\dot\Lambda_0+\dot\Lambda_1-2\dot\delta$. Moreover, using (9) and (11), we see that for all $|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle=|\boldsymbol{\lambda}_n,\boldsymbol{s}_n\rangle^\bullet\in\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot w\rangle$, we have $N_0(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=2$, $N_1(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=1$, $N_2(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=0$ and $N_0(\boldsymbol{\lambda}_n;\boldsymbol{s}_n;l)=N_1(\boldsymbol{\lambda}_n;\boldsymbol{s}_n;l)=0$ (this shows *a posteriori* that $\dim(\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle)=1$). $\diamond$

2.4 Action of the Weyl groups W_n and $\dot W_l$

The Weyl group W_n acts on the weight lattice $\bigoplus_{i=0}^{n-1}\mathbb{Z}\Lambda_i\oplus\mathbb{Z}\delta$ by

$$(17)\quad \sigma_i.\delta=\delta \quad\text{and}\quad \sigma_i.\Lambda_j=\begin{cases}\Lambda_j & \text{if } j\neq i,\\ \Lambda_{i-1}+\Lambda_{i+1}-\Lambda_i-\delta_{i,0}\,\delta & \text{if } j=i\end{cases}\qquad (0\le i,j\le n-1).$$

Moreover, it is easy to see that W_n acts faithfully on $\mathbb{Z}^n(s)$ by

$$(18)\qquad \begin{cases}\sigma_0.(s_1,\dots,s_n) &=& (s_n+l,s_2,\dots,s_{n-1},s_1-l),\\ \sigma_i.(s_1,\dots,s_n) &=& (s_1,\dots,s_{i+1},s_i,\dots,s_n)\qquad (1\le i\le n-1),\end{cases}$$

and the set $A_{n,l}(s)$ defined by (14) is a fundamental domain for this action. In a similar way, one can define two actions of the Weyl group $\dot W_l$ of $U_p(\widehat{\mathfrak{sl}}_l)$, one on the weight lattice $\bigoplus_{i=0}^{l-1}\mathbb{Z}\dot\Lambda_i\oplus\mathbb{Z}\dot\delta$ and one on $\mathbb{Z}^l(s)$.

2.5 The lower crystal basis $(\mathcal{L}[\boldsymbol{s}_l],\mathcal{B}[\boldsymbol{s}_l])$ of $\mathbf{F}_q[\boldsymbol{s}_l]$ at $q=0$

Let $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$. Following [Kas1], let $\mathbb{A}\subset\mathbb{Q}(q)$ be the ring of rational functions which are regular at $q=0$, $\mathcal{L}[\boldsymbol{s}_l]:=\bigoplus_{\boldsymbol{\lambda}_l\in\Pi^l}\mathbb{A}\,|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle$ and for $0\le i\le n-1$, let $\tilde e_i=\tilde e_i^{\,\mathrm{low}}$ and $\tilde f_i=\tilde f_i^{\,\mathrm{low}}$ denote Kashiwara's operators acting on $\mathcal{L}[\boldsymbol{s}_l]$. Put

$$(19)\qquad \mathcal{B}[\boldsymbol{s}_l]:=\{|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle \bmod q\mathcal{L}[\boldsymbol{s}_l]\mid \boldsymbol{\lambda}_l\in\Pi^l\}.$$

In the sequel, if $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$ is fixed, we shall write more briefly $\boldsymbol{\lambda}_l$ for the element in $\mathcal{B}[\boldsymbol{s}_l]$ indexed by the corresponding multi-partition. By [JMMO],[FLOTW],[U], the pair $(\mathcal{L}[\boldsymbol{s}_l],\mathcal{B}[\boldsymbol{s}_l])$ is a lower crystal basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ at $q=0$ in the sense of [Kas1], and the crystal graph $\mathcal{B}[\boldsymbol{s}_l]$ contains the arrow $\boldsymbol{\lambda}_l\xrightarrow{i}\boldsymbol{\mu}_l$ if and only if the multi-partition $\boldsymbol{\mu}_l$ is obtained from $\boldsymbol{\lambda}_l$ by adding a good i-node in the sense of [U, Thm. 2.4].

We now recall the definition of the involution σ_i of $\mathcal{B}[\boldsymbol{s}_l]$ (we sometimes view σ_i as a bijection of Π^l). First, let us introduce a piece of notation that will be used in the sequel.

Notation 2.7 For $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$ and $w\in\mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, put

$$(20)\qquad \Pi^l(\boldsymbol{s}_l;w):=\{\boldsymbol{\lambda}_l\in\Pi^l\;\big|\;|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle\in\Lambda^s\langle w\rangle\},$$

and define similarly $\Pi^n(\boldsymbol{s}_n;\dot w)$ for $\boldsymbol{s}_n\in\mathbb{Z}^n(s)$ and $\dot w\in\dot{\mathcal{P}}(\mathbf{F}_p[\boldsymbol{s}_n]^\bullet)$. $\diamond$

Definition 2.8 Fix $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Let $\boldsymbol{\lambda}_l \in \mathcal{B}[\boldsymbol{s}_l] \cong \Pi^l$ and $i \in [\![0; n-1]\!]$. Let $\mathcal{C}$ be the i-chain in $\mathcal{B}[\boldsymbol{s}_l]$ containing $\boldsymbol{\lambda}_l$. Let $\sigma_i(\boldsymbol{\lambda}_l) \in \mathcal{B}[\boldsymbol{s}_l] \cong \Pi^l$ be the unique element in $\mathcal{C}$ such that $\mathrm{wt}(\sigma_i(\boldsymbol{\lambda}_l)) = \sigma_i.(\mathrm{wt}(\boldsymbol{\lambda}_l))$. In other words, $\sigma_i(\boldsymbol{\lambda}_l)$ is obtained from $\boldsymbol{\lambda}_l$ via a central symmetry in the middle of $\mathcal{C}$. This defines an involution σ_i of $\mathcal{B}[\boldsymbol{s}_l]$. This map induces, for $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, a bijection

$$\sigma_i : \Pi^l(\boldsymbol{s}_l; w) \xrightarrow{\sim} \Pi^l(\boldsymbol{s}_l; \sigma_i.w). \tag{21}$$

$\diamond$

By [Kas2], the definition of $\sigma_0, \dots, \sigma_{n-1}$ as bijections of $\mathcal{B}[\boldsymbol{s}_l]$ gives actually rise to an action of the Weyl group W_n on $\mathcal{B}[\boldsymbol{s}_l]$, but we do not need this fact in the sequel. The next proposition gives a simple expression for $\sigma_i(\boldsymbol{\lambda}_l)$ when $\boldsymbol{\lambda}_l$ is located at the head of an i-chain in $\mathcal{B}[\boldsymbol{s}_l]$.

Proposition 2.9 *Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $i \in [\![0; n-1]\!]$ be such that $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. Let $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w)$ and $\boldsymbol{\mu}_l := \sigma_i(\boldsymbol{\lambda}_l)$. Then*

(i) *$\boldsymbol{\mu}_l$ is the multi-partition obtained by adding to $\boldsymbol{\lambda}_l$ all its addable i-nodes, and there are $k_i = (w, \alpha_i)$ of them.*

(ii) *We have $|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle = f_i^{(k_i)}.|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle$ and $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = e_i^{(k_i)}.|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle$.*

Proof. See [Y2, Prop. 3.5]. $\square$

2.6 Uglov's canonical bases of the Fock spaces

Following [U], the space Λ^s can be endowed with an involution $\overline{}$. Instead of recalling the definition of this involution, we give its main properties. They turn out to characterize it completely; see Remark 3.12.

Proposition 2.10 ([U]) *There exists an involution $\overline{}$ of Λ^s such that:*

(i) *$\overline{}$ is a $\mathbb{Q}$-linear map of Λ^s such that for all $u \in \Lambda^s$, $k \in \mathbb{Z}$, we have $\overline{q^k\, u} = q^{-k}\, \overline{u}$.*

(ii) (Unitriangularity property.) *For all $\lambda \in \Pi$, we have*

$$\overline{|\lambda, s\rangle} \in |\lambda, s\rangle + \bigoplus_{\mu \lhd \lambda} \mathbb{Z}[q, q^{-1}]\, |\mu, s\rangle,$$

where $\lhd$ stands for the dominance ordering on partitions.

(iii) *For all $\lambda \in \Pi$, we have $\mathrm{wt}(\overline{|\lambda, s\rangle}) = \mathrm{wt}(|\lambda, s\rangle)$ and $\dot{\mathrm{wt}}(\overline{|\lambda, s\rangle}) = \dot{\mathrm{wt}}(|\lambda, s\rangle)$.*

(iv) *For all $0 \le i \le n-1$, $0 \le j \le l-1$, $m < 0$, $v \in \Lambda^s$, we have*

$$\overline{f_i.v} = f_i.\overline{v}, \quad \overline{\dot{f}_j.v} = \dot{f}_j.\overline{v} \quad \textit{and} \quad \overline{B_m.v} = B_m.\overline{v}.$$

$\square$

By [U], this involution can be computed by straightening non-ordered q-wedge products. However, the number of q-wedge products of 2 factors to be straightened is in general too large for practical computations. We shall derive in Section 3 another more efficient algorithm that does not require straightening q-wedge products.

By Propositions 2.10 (iii) and 2.4, the higher-level Fock spaces are stable under the involution $\overline{}$. The involution induced on these spaces will still be denoted by $\overline{}$. Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. For $\boldsymbol{\mu}_l \in \Pi^l$, write

$$\overline{|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle} = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \tag{22}$$

with $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q) \in \mathbb{Z}[q, q^{-1}]$, and let

$$A_{\boldsymbol{s}_l}(q) := \big(a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\big)_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l} \tag{23}$$

denote the matrix of the involution $\overline{}$ of $\mathbf{F}_q[\boldsymbol{s}_l]$ with respect to the standard basis. Since the weight subspaces of $\mathbf{F}_q[\boldsymbol{s}_l]$ are stable under the involution $\overline{}$, (9) implies that $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)$ is zero unless $|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l|$, where $|\boldsymbol{\lambda}_l|$ (*resp.* $|\boldsymbol{\mu}_l|$) denotes the number of boxes contained in the Young diagram of $\boldsymbol{\lambda}_l$ (*resp.* $\boldsymbol{\mu}_l$). By Proposition 2.10 (ii), the matrix $A_{\boldsymbol{s}_l}(q)$ is unitriangular. As a consequence, one can define, by a classical argument, canonical bases as follows.

Theorem 2.11 ([U]) *Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Then there exists a unique basis*

$$\{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\} \qquad \Big(\text{resp. } \{G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}\Big)$$

of $\mathbf{F}_q[\boldsymbol{s}_l]$ such that:

(i) $\overline{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)} = G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ (resp. $\overline{G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)} = G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$),
(ii) $G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \equiv |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \mod q\mathcal{L}^+[\boldsymbol{s}_l]$ (resp. $G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \equiv |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \mod q^{-1}\mathcal{L}^-[\boldsymbol{s}_l]$),

where $\mathcal{L}^\epsilon[\boldsymbol{s}_l] := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{Z}[q^\epsilon]\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle$ $(\epsilon = \pm 1)$. □

For $\epsilon = \pm 1$, define entries $\Delta^\epsilon_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q) \in \mathbb{Z}[q, q^{-1}]$ ($\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l \in \Pi^l$) by

$$G^\epsilon(\boldsymbol{\mu}_l, \boldsymbol{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} \Delta^\epsilon_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle, \tag{24}$$

and denote by

$$\Delta^\epsilon_{\boldsymbol{s}_l}(q) := \big(\Delta^\epsilon_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\big)_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l} \qquad (\epsilon = \pm 1) \tag{25}$$

the transition matrices between the standard and the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]$.

We shall give some (parts of the) canonical bases of different Fock spaces in Section 4. By [U], the entries of $\Delta^+_{\boldsymbol{s}_l}(q)$ (*resp.* $\Delta^-_{\boldsymbol{s}_l}(q)$) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A, so by [KT], these polynomials are in $\mathbb{N}[q]$ (*resp.* $\mathbb{N}[p]$). Moreover, both canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]$ are dual to each other with respect to a certain bilinear form, which gives an inversion formula for Kazhdan-Lusztig polynomials; see [U, Thm. 5.15]. By [U], the basis $\{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is a lower global crystal basis (in the sense of [Kas1]) of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{F}_q[\boldsymbol{s}_l]$.

3 Computation of the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]$

3.1 A $^-$-invariant basis $\mathcal{B}$ of Λ^s

Notation 3.1 In this section, we use the following notation.

* In this article, we always identify the multi-partition $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi^l$ with its Young diagram $\{(i, j, b) \in \mathbb{N}^* \times \mathbb{N}^* \times [\![1; l]\!] \mid 1 \leq j \leq \lambda^{(b)}_i\}$. Let $(i(\gamma), j(\gamma), b(\gamma))$ denote the coordinates of the node $\gamma \in \boldsymbol{\lambda}_l$. Let

  $$\partial\boldsymbol{\lambda}_l := \{(i, j, b) \in \boldsymbol{\lambda}_l \mid j = \lambda^{(b)}_i\} \tag{26}$$

  denote the *border* of $\boldsymbol{\lambda}_l$, that is the vertical strip made of the rightmost nodes of $\boldsymbol{\lambda}_l$.

* For $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, let

  $$\mathbf{M}_q[\boldsymbol{s}_l] := U_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle. \tag{27}$$

  Let $\Pi^l(\boldsymbol{s}_l)^\circ = \Pi^l(\boldsymbol{s}_l, n)^\circ$ denote the set of l-multi-partitions indexing the crystal graph of the $U_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{M}_q[\boldsymbol{s}_l]$. For $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, put

  $$\Pi^l(\boldsymbol{s}_l; w)^\circ = \Pi^l(\boldsymbol{s}_l, n; w)^\circ := \Pi^l(\boldsymbol{s}_l; w) \cap \Pi^l(\boldsymbol{s}_l)^\circ. \tag{28}$$

* Let

  $$\mathcal{X}_{l,n} := \{(v_1, \dots, v_l) \in \mathbb{Z}^l \mid n-1 \geq v_1 \geq \cdots \geq v_l \geq 0\}. \tag{29}$$

  Note that the $U'_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{M}_q[\boldsymbol{s}_l]$ $\big(\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l(s)\big)$ is isomorphic to $\mathbf{M}_q[\boldsymbol{v}_l]$ for a unique multi-charge $\boldsymbol{v}_l = (v_1, \dots, v_l) \in \mathcal{X}_{l,n}$; more precisely, this $\boldsymbol{v}_l$ is related to $\boldsymbol{s}_l$ by $\Lambda_{v_1} + \cdots + \Lambda_{v_l} = \Lambda_{s_1} + \cdots + \Lambda_{s_l}$. $\diamond$

3.1.1 A basis of $\mathbf{M}_q[\boldsymbol{s}_l]$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$)

In [J1], Jacon gave a monomial basis of $\mathbf{M}_q[\boldsymbol{v}_l]$ with $\boldsymbol{v}_l \in \mathcal{X}_{l,n}$. Using this basis, he derived in [J2] an algorithm for computing the canonical bases of $\mathbf{M}_q[\boldsymbol{v}_l]$. We recall here his result and give a basis of the same type for $\mathbf{M}_q[\boldsymbol{s}_l]$ with $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$.

Let $\boldsymbol{v}_l \in \mathcal{X}_{l,n}$, $\boldsymbol{\lambda}_l \in \Pi^l$ and $0 \leq k \leq n-1$. Let X_k be the set of all removable k-nodes γ in $\boldsymbol{\lambda}_l$ such that for every $(k-1)$-node β in $\partial\boldsymbol{\lambda}_l$, we have $j(\gamma) > j(\beta)$. Here in the definition of i-nodes ($i = k$ or $k-1$), the residues are taken with respect to the multi-charge $\boldsymbol{v}_l$.

Example 3.2 Take $n = 4$, $l = 2$, $\boldsymbol{v}_l = (3,1)$ and $\boldsymbol{\lambda}_l = ((4,2),(4,1))$. Then we have

$$X_0 = \{(1,4,2)\}, \quad X_1 = \emptyset, \quad X_2 = \{(1,4,1)\} \quad \text{and} \quad X_3 = \emptyset.$$

◇

Jacon proved the following result.

Proposition 3.3 ([J1], Lemmas 4.2 & 4.3) *Let $\boldsymbol{v}_l \in \mathcal{X}_{l,n}$ and $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi^l(\boldsymbol{v}_l)^\circ$, $\boldsymbol{\lambda}_l \neq \boldsymbol{\emptyset}_l$. Let $j_{\max} := \max\{\lambda_1^{(b)} \mid 1 \le b \le l\}$ denote the largest part of $\boldsymbol{\lambda}_l$ and for $0 \le k \le n-1$, let X_k be the set defined above. Then there exists $k \in [\![0; n-1]\!]$ such that*

$$X_k \cap \{\gamma \in \boldsymbol{\lambda}_l \mid j(\gamma) = j_{\max}\}$$

is nonempty. Moreover, the multi-partition obtained from $\boldsymbol{\lambda}_l$ by removing all the nodes in X_k (for k as above) is in $\Pi^l(\boldsymbol{v}_l)^\circ$. □

Thanks to this proposition, one can define recursively a Jacon element $F(\boldsymbol{\lambda}_l) \in U_q(\widehat{\mathfrak{sl}}_n)$.

Definition 3.4 Let $\boldsymbol{v}_l \in \mathcal{X}_{l,n}$ and $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l)^\circ$. If $\boldsymbol{\lambda}_l$ is the empty multi-partition, let $F(\boldsymbol{\lambda}_l) := 1 \in U_q(\widehat{\mathfrak{sl}}_n)$. Otherwise, let $k \in [\![0; n-1]\!]$ be the minimal integer given by Proposition 3.3 and let X_k be corresponding set of removable k-nodes. Then $\boldsymbol{\mu}_l := \boldsymbol{\lambda}_l \setminus X_k$ is in $\Pi^l(\boldsymbol{v}_l)^\circ$, so by induction, we can define

$$F(\boldsymbol{\lambda}_l) := f_k^{(\sharp X_k)} F(\boldsymbol{\mu}_l). \tag{30}$$

Note that $F(\boldsymbol{\lambda}_l)$ is a product of divided powers of the f_k's. ◇

Example 3.5 Take $n = 4$, $l = 2$, $\boldsymbol{v}_l = (3,1) \in \mathcal{X}_{l,n}$ and $\boldsymbol{\lambda}_l = ((4,2),(4,1))$. One can check, *e.g.* by computing the crystal graph of $\mathbf{M}_q[\boldsymbol{v}_l]$, that $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l)^\circ$. We have

$$F(\boldsymbol{\lambda}_l) = f_0^{(1)} f_2^{(1)} f_1^{(1)} f_3^{(2)} f_0^{(2)} f_2^{(2)} f_1^{(1)} f_3^{(1)}.$$

◇

Jacon proved the following result.

Theorem 3.6 ([J1], Prop. 4.6) *Let $\boldsymbol{v}_l \in \mathcal{X}_{l,n}$ and $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l)^\circ$. Then*

$$F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{v}_l\rangle \in |\boldsymbol{\lambda}_l, \boldsymbol{v}_l\rangle + \bigoplus_{\substack{\boldsymbol{\mu}_l \in \Pi^l \\ a(\boldsymbol{\mu}_l) > a(\boldsymbol{\lambda}_l)}} \mathbb{Z}[q, q^{-1}]\, |\boldsymbol{\mu}_l, \boldsymbol{v}_l\rangle,$$

where $a : \Pi^l \to \mathbb{Z}$ denotes Lusztig's a-value [BK, J1]. □

We do not use this a-value in the sequel, but only the unitriangularity property of this theorem, which implies the following.

Corollary 3.7 *Let* $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l(s)$*, and* $\boldsymbol{v}_l = (v_1, \dots, v_l)$ *be the unique element in* $\mathcal{X}_{l,n}$ *such that* $\Lambda_{v_1} + \cdots + \Lambda_{v_l} = \Lambda_{s_1} + \cdots + \Lambda_{s_l}$*. For* $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l)^\circ$*, let* $F(\boldsymbol{\lambda}_l)$ *be the Jacon element associated to* $\boldsymbol{\lambda}_l$ *and* $\boldsymbol{v}_l$*. Then*

$$\{F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l)^\circ\}$$

is a basis of $\mathbf{M}_q[\boldsymbol{s}_l]$*.*

Proof. Fix a weight w of $\mathbf{M}_q[\boldsymbol{v}_l]$. Let $B_w := \{F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{v}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l; w)^\circ\}$. By Theorem 3.6, the transition matrix between the standard basis of $\mathbf{M}_q[\boldsymbol{v}_l]\langle w\rangle$ and B_w is unitriangular with respect to a certain ordering given by the a-value, which shows that B_w is a basis of $\mathbf{M}_q[\boldsymbol{v}_l]\langle w\rangle$. This proves the corollary if $\boldsymbol{s}_l = \boldsymbol{v}_l$. Now, by definition of $\boldsymbol{v}_l$, there exists an isomorphism $\mathbf{M}_q[\boldsymbol{v}_l] \xrightarrow{\sim} \mathbf{M}_q[\boldsymbol{s}_l]$ of $U'_q(\widehat{\mathfrak{sl}}_n)$-modules that maps $u.|\boldsymbol{\emptyset}_l, \boldsymbol{v}_l\rangle$ on $u.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$ $(u \in U'_q(\widehat{\mathfrak{sl}}_n))$. The result follows. $\square$

3.1.2 A basis of $\mathcal{H}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$)

Recall that the operators B_m, $m > 0$ (*resp.* $m < 0$) pairwise commute. For any partition $\lambda = (\lambda_1, \dots, \lambda_r)$, put

$$B_\lambda := B_{\lambda_1} \cdots B_{\lambda_r} \in \mathrm{End}(\Lambda^s) \tag{31}$$

(by convention, $B_\emptyset$ is the identity operator), and define in a similar way $B_{-\lambda}$.

Proposition 3.8 *Let* $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$*. Then* $\{B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle \mid \lambda \in \Pi\}$ *is a basis of* $\mathcal{H}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$*.*

Proof. Let Sym denote the $\mathbb{Q}(q)$-vector space of symmetric functions and let $\{p_\lambda \mid \lambda \in \Pi\}$ be the basis formed by the power sums [Mac]. Let us recall the action of $\mathcal{H}$ on Sym. For $k < 0$ let b_k denote the multiplication in Sym by $p_{(-k)}$. For $k > 0$ and $f = f(p_{(1)}, p_{(2)}, \dots) \in \mathrm{Sym}$, put $b_k(f) := \gamma_k \frac{\partial f}{\partial p_{(k)}}$, where $\gamma_k \in \mathcal{Z}(\mathcal{H}) \cong \mathbb{Q}(q)$ is the scalar such that $[B_k, B_{-k}] = \gamma_k$. One easily checks that there exists a homomorphism of algebras $\mathcal{H} \to \mathrm{End}(\mathrm{Sym})$ that maps B_k on b_k $(k \in \mathbb{Z}^*)$. This makes Sym into a simple $\mathcal{H}$-module. Note that the vector $v := |\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle \in \mathbf{F}_q[\boldsymbol{s}_l]$ is a singular vector for the action of $\mathcal{H}$, that is, $v \neq 0$ and $B_k.v = 0$ for all $k > 0$. By a classical result (see *e.g.* [Kac, Lemma 9.13 a)]), the linear map: $\mathrm{Sym} \to \mathcal{H}.v$, $p_\lambda \mapsto B_{-\lambda}.v$ $(\lambda \in \Pi)$, is an isomorphism of $\mathcal{H}$-modules. Since $\{p_\lambda \mid \lambda \in \Pi\}$ is a basis of Sym, the result follows. $\square$

3.1.3 A $^-$-invariant basis $\mathcal{B}$ of Λ^s

We need a lemma on bimodules, whose proof is left to the reader.

Lemma 3.9 *Let* A *be* $\mathbb{Q}(q)$*-algebra with unity, and* $U := U'_q(\widehat{\mathfrak{sl}}_n)$ *or* $U'_p(\widehat{\mathfrak{sl}}_l)$*. Let* V *be an* $(A \otimes U)$*-module such that the actions of* $A \cong (A \otimes 1)$ *and* $U \cong (1 \otimes U)$ *commute. Assume that* V *is an integrable* U*-module and there exists a highest weight vector (for the action of* U*), denoted by* v_0*, such that* $V = (A \otimes U).v_0 = A.U.v_0$*. Let* $A^\bullet \subset A$ *and* $U^\bullet \subset U$ *be subsets*

such that $\{a.v_0 \mid \ a \in A^\bullet\}$ *is a basis of* $A.v_0$ *and* $\{u.v_0 \mid \ u \in U^\bullet\}$ *is a basis of* $U.v_0$*. Then* $\{a.u.v_0 \mid a \in A^\bullet,\ u \in U^\bullet\}$ *is a basis of* V*.* $\square$

Recall Notation 3.1 and the definition of the Jacon elements in $U'_q(\widehat{\mathfrak{sl}}_n)$. In a similar way, define a Jacon element $\dot{F}(\boldsymbol{\lambda}_n) \in U'_p(\widehat{\mathfrak{sl}}_l)$ for $\boldsymbol{\lambda}_n \in \Pi^l(\boldsymbol{v}_n, l)^\circ$, $\boldsymbol{v}_n \in \mathcal{X}_{n,l}$.

Lemma 3.10 *Keep the notation above. Let* $\boldsymbol{r}_l \in A_{l,n}(s)$*. By [U], there exists a unique* $\boldsymbol{r}_n \in A_{n,l}(s)$ *such that* $|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = |\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$*. Let* $\boldsymbol{v}_l$ *be the unique multi-charge in* $\mathcal{X}_{l,n}$ *such that* $U'_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{v}_l\rangle \cong U'_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$ *and let* $\boldsymbol{v}_n$ *be the unique multi-charge in* $\mathcal{X}_{n,l}$ *such that* $U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{v}_n\rangle^\bullet \cong U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet$*. Then*

$$\begin{aligned}
&\{F(\boldsymbol{\lambda}_l)B_{-\mu}\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\} \\
&= \{\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\} \\
&= \{F(\boldsymbol{\lambda}_l)\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\}
\end{aligned}$$

is a basis of the vector space

$$U'_q(\widehat{\mathfrak{sl}}_n)\otimes\mathcal{H}\otimes U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = U'_p(\widehat{\mathfrak{sl}}_l)\otimes\mathcal{H}\otimes U'_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = U'_q(\widehat{\mathfrak{sl}}_n)\otimes U'_p(\widehat{\mathfrak{sl}}_l)\otimes\mathcal{H}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle.$$

Proof. All the equalities come from the fact that the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ pairwise commute. By Corollary 3.7 and Proposition 3.8,

- $\{F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ\}$ is a basis of $U'_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$,
- $\{B_{-\mu}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \mu \in \Pi\}$ is a basis of $\mathcal{H}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$, and
- $\{\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet \mid \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\}$ is a basis of $U'_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet$.

We now conclude with Lemma 3.9. $\square$

We are now ready to state the following:

Theorem 3.11 *With notation of Lemma 3.10, the set*

$$\begin{aligned}
\mathcal{B} &:= \{F(\boldsymbol{\lambda}_l)B_{-\mu}\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet \mid \boldsymbol{r}_n \in A_{n,l}(s),\ \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\} \\
&= \{\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{r}_l \in A_{l,n}(s),\ \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\} \\
&= \{F(\boldsymbol{\lambda}_l)\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{r}_l \in A_{l,n}(s),\ \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ,\ \mu \in \Pi,\ \boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ\}
\end{aligned}$$

is a basis of Λ^s *that is* $^-$ *-invariant.*

Proof. The fact that $\mathcal{B}$ is a basis of Λ^s comes from Lemma 3.10 and Theorem 2.3. Let $\boldsymbol{r}_l \in \mathbb{Z}^l(s)$. By (9), the subspace of $\mathbf{F}_q[\boldsymbol{r}_l]$ of weight $\mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle)$ is one-dimensional. Therefore, by Proposition 2.10 (ii), the vector $|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$ is $^-$ -invariant. Proposition 2.10 (iv) then implies that every vector in $\mathcal{B}$ is $^-$ -invariant. $\square$

Remark 3.12 The proof that $\mathcal{B}$ is a basis of Λ^s does not use the involution $^-$. Since every vector in $\mathcal{B}$ is fixed by the involution $^-$, this determines this involution completely. $\diamond$

3.2 Computation of $\mathcal{B}\cap \mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$ ($\boldsymbol{s}_l\in\mathbb{Z}^l(s)$, $v\in\mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$)

Let $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$ and v be a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. Let $\mathcal{B}$ be the $^-$ -invariant basis of Λ^s given by Theorem 3.11. The goal of this section is to compute the list of vectors of $\mathcal{B}$ that lie in $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$. Since the weights of the operators $F(\boldsymbol{\lambda}_l)$, $B_{-\mu}$ and $\dot F(\boldsymbol{\lambda}_n)$ from Theorem 3.11 are known, it is enough to determine whether the sum of these weights and the weight of $|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l\rangle$ is equal to v. This raises no theoretical problem but requires, in view of practical computations, to introduce some cumbersome notation.

Define a partial ordering on $P:=\bigoplus_{i=0}^{n-1}\mathbb{Z}\Lambda_i\oplus\mathbb{Z}\delta$ by writing $w'\le w''$ (w', $w''\in P$) if $w''-w'\in\sum_{i=0}^{n-1}\mathbb{N}\alpha_i$.

Definition 3.13 We say that the weight $w\in P$ is *admissible* (with respect to $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$) if:

(i) w is a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$ such that $v\le w$.

(ii) Let $(\boldsymbol{r}_n,\dot w)\in\mathbb{Z}^n(s)\times\dot{\mathcal{P}}(\Lambda^s)$ be the unique pair such that $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot w\rangle$ (see Proposition 2.5). We then require $\boldsymbol{r}_n$ to be in $A_{n,l}(s)$. $\diamond$

Notation 3.14 Let w be an admissible weight with respect to $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$. Then by Definition 3.13, there exist $N_0,\dots,N_{n-1}\in\mathbb{N}$ such that $w-v=\sum_{i=0}^{n-1}N_i\,\alpha_i$. Let $N(w):=\min_i N_i$. The pair $(\boldsymbol{r}_n,\dot w)\in A_{n,l}(s)\times\dot{\mathcal{P}}(\Lambda^s)$ such that $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot w\rangle$ will be denoted by $\big(\boldsymbol{r}_n(w),\dot w(w)\big)$. Let $\boldsymbol{r}_l(w)\in A_{l,n}(s)$ be such that $|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l(w)\rangle=|\boldsymbol{\emptyset}_n,\boldsymbol{r}_n(w)\rangle^\bullet$. Recall the notation $\mathcal{X}_{l,n}$ and $\mathcal{X}_{n,l}$ from (29). Then there exists a unique $\boldsymbol{v}_l(w)\in\mathcal{X}_{l,n}$ (*resp.* $\boldsymbol{v}_n(w)\in\mathcal{X}_{n,l}$) such that $U_q'(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l,\boldsymbol{v}_l(w)\rangle$ is isomorphic to $U_q'(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l(w)\rangle$ (*resp.* $U_p'(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n,\boldsymbol{v}_n(w)\rangle^\bullet$ is isomorphic to $U_p'(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n,\boldsymbol{r}_n(w)\rangle^\bullet$). $\diamond$

Example 3.15 Take $n=3$, $l=2$, $\boldsymbol{s}_l=(3,6)$ and $v=2\Lambda_0-4\delta=\mathrm{wt}(|\boldsymbol{\emptyset}_l,\boldsymbol{s}_l\rangle)-(\alpha_0+\alpha_1+\alpha_2)$. The following array gives the list of the admissible weights w with respect to $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$ and the corresponding $N(w)$, $\dot w(w)$, $\boldsymbol{r}_n(w)$, $\boldsymbol{v}_n(w)$, $\boldsymbol{r}_l(w)$ and $\boldsymbol{v}_l(w)$.

w	$N(w)$	$\dot w(w)$	$\boldsymbol{r}_n(w)$	$\boldsymbol{v}_n(w)$	$\boldsymbol{r}_l(w)$	$\boldsymbol{v}_l(w)$
$2\Lambda_0-3\delta$	1	$3\dot\Lambda_1-3\dot\alpha_1$	$(3,3,3)$	$(1,1,1)$	$(6,3)$	$(0,0)$
$(2\Lambda_0-3\delta)-\alpha_0$	0	$2\dot\Lambda_0+\dot\Lambda_1-(\dot\alpha_0+3\dot\alpha_1)$	$(4,3,2)$	$(1,0,0)$	$(5,4)$	$(2,1)$
$(2\Lambda_0-3\delta)-(\alpha_0+\alpha_1+\alpha_2)$	0	$3\dot\Lambda_1-(\dot\alpha_0+4\dot\alpha_1)$	$(3,3,3)$	$(1,1,1)$	$(6,3)$	$(0,0)$

$\diamond$

Proposition 3.16 *Let $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$ and v be a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. Let $\mathcal{B}$ be the $^-$ -invariant basis of Λ^s given by Theorem 3.11. Put $\mathbb{B}:=\mathcal{B}\cap\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$. Then with Notation 3.14, we have*

$$\mathbb{B}=\bigcup_{w\ \textit{admissible}}\Big\{F(\boldsymbol{\lambda}_l)\dot F(\boldsymbol{\lambda}_n)B_{-\mu}.|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l(w)\rangle\ \Big|\ \mu\in\Pi,\ |\mu|\le N(w),$$
$$\boldsymbol{\lambda}_n\in\Pi^n(\boldsymbol{v}_n(w),l)^\circ,\ \dot{\mathrm{wt}}\big(\dot F(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n,\boldsymbol{r}_n(w)\rangle^\bullet\big)=\dot w(w),$$
$$\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{v}_l(w),n)^\circ,\ \mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l(w)\rangle\big)=w'(w,\mu)\Big\},$$

where we put $w'(w,\mu):=v-w+|\mu|\delta+\mathrm{wt}(|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l(w)\rangle)$.

Proof.

* Proof of inclusion $\supset$. Let $x := F(\boldsymbol{\lambda}_l)\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle$ with w admissible and $\boldsymbol{\lambda}_l$, μ and $\boldsymbol{\lambda}_n$ as in the right hand-side of the statement of this proposition. We must show that $x \in \mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$. Put $y := \dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet$. Since $\mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) = w'(w,\mu)$ and the operator $B_{-\mu}$ has weight $-|\mu|\delta$, we have
$$\begin{aligned}\mathrm{wt}(x) &= \mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) + \mathrm{wt}(y) - \mathrm{wt}\big(|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet\big) - |\mu|\delta \\ &= w'(w,\mu) + \mathrm{wt}(y) - \mathrm{wt}\big(|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) - |\mu|\delta \\ &= v - w + \mathrm{wt}(y).\end{aligned}$$
By assumption, we have $\dot{\mathrm{wt}}(y) = \dot{w}(w)$, whence $y \in \mathbf{F}_p[\boldsymbol{r}_n(w)]^\bullet\langle \dot{w}(w)\rangle = \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$. This implies $\mathrm{wt}(y) = w$ and $\mathrm{wt}(x) = v$. Moreover, since the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ pairwise commute, we have
$$x = F(\boldsymbol{\lambda}_l)B_{-\mu}.y \;\in\; U'_p(\widehat{\mathfrak{sl}}_l).\mathcal{H}\left(\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle\right) \;\subset\; \mathbf{F}_q[\boldsymbol{s}_l].$$

* Proof of inclusion $\subset$. Let $x \in \mathbb{B}$. With notation from Theorem 3.11, let $\boldsymbol{r}_n \in A_{n,l}(s)$, $\boldsymbol{r}_l \in A_{l,n}(s)$, $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l, n)^\circ$, $\mu \in \Pi$ and $\boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n, l)^\circ$ be such that
$$x = F(\boldsymbol{\lambda}_l)B_{-\mu}\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = F(\boldsymbol{\lambda}_l)\dot{F}(\boldsymbol{\lambda}_n)B_{-\mu}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle.$$
Since $x \neq 0$, the vector $y := \dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = \dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$ is nonzero, so it is a weight vector of the $U_q(\widehat{\mathfrak{sl}}_n)$-module Λ^s. Let us now show that $w := \mathrm{wt}(y)$ is an admissible weight. Let $\boldsymbol{t}_l \in \mathbb{Z}^l(s)$ be such that $y \in \mathbf{F}_q[\boldsymbol{t}_l]$. Then we have
$$x = F(\boldsymbol{\lambda}_l)B_{-\mu}.y \in \mathbf{F}_q[\boldsymbol{t}_l]\big\langle\, w - |\mu|\delta - \textstyle\sum_{i=0}^{n-1} n_i\,\alpha_i\big\rangle,$$
where for $0 \leq i \leq n-1$ we put $n_i := N_i(\boldsymbol{\lambda}_l; \boldsymbol{v}_l; n)$. Moreover, since $x \in \mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$ and the spaces $\mathbf{F}_q[\boldsymbol{a}_l]$, $\boldsymbol{a}_l \in \mathbb{Z}^l(s)$ are in direct sum, we must have $\boldsymbol{t}_l = \boldsymbol{s}_l$ and
$$(*): \quad v = w - |\mu|\delta - \sum_{i=0}^{n-1} n_i\,\alpha_i = w - \sum_{i=0}^{n-1}(n_i + |\mu|)\,\alpha_i.$$
In particular, we have $y \in \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ and $v \leq w$. By Proposition 2.5, there exists a unique pair $(\boldsymbol{s}_n, \dot{w}) \in \mathbb{Z}^n(s) \times \dot{\mathcal{P}}(\Lambda^s)$ such that $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle \dot{w}\rangle = \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$. We thus have
$$y = \dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet \in \mathbf{F}_p[\boldsymbol{r}_n]^\bullet \cap \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle = \mathbf{F}_p[\boldsymbol{r}_n]^\bullet \cap \mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle \dot{w}\rangle \subset \mathbf{F}_p[\boldsymbol{r}_n]^\bullet \cap \mathbf{F}_p[\boldsymbol{s}_n]^\bullet.$$
Since the $\mathbf{F}_p[\boldsymbol{a}_n]^\bullet$, $\boldsymbol{a}_n \in \mathbb{Z}^n(s)$ are in direct sum, we must have $\boldsymbol{s}_n = \boldsymbol{r}_n \in A_{n,l}(s)$. As a consequence, w is an admissible weight. One easily checks that $\dot{w} = \dot{w}(w)$, $\boldsymbol{r}_n = \boldsymbol{r}_n(w)$, $\boldsymbol{v}_n = \boldsymbol{v}_n(w)$, $\boldsymbol{r}_l = \boldsymbol{r}_l(w)$ and $\boldsymbol{v}_l = \boldsymbol{v}_l(w)$. By $(*)$ we have $w - v = \sum_{i=0}^{n-1} N_i\,\alpha_i$ with $N_i := n_i + |\mu|$ $(0 \leq i \leq n-1)$. Since the n_i's are nonnegative integers, we must have $|\mu| = N_i - n_i \leq N_i$ for all $0 \leq i \leq n-1$, whence $|\mu| \leq \min_i(N_i) = N(w)$. Since $y \in \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle = \mathbf{F}_p[\boldsymbol{r}_n(w)]^\bullet\langle \dot{w}(w)\rangle$, we have $\dot{\mathrm{wt}}\big(\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet\big) = \dot{w}(w)$. Finally, we have
$$v = \mathrm{wt}(x) = \mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) + \mathrm{wt}(y) - \mathrm{wt}\big(|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet\big) - |\mu|\delta$$
and $\mathrm{wt}(y) = w$, which implies $\mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) = w'(w,\mu)$. □

Let us now explain how we can apply Proposition 3.16. Let w be an admissible weight with respect to $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$. We compute the list of the multi-partitions $\boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n(w), l)^\circ$ such that $\dot{\mathrm{wt}}\big(\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet\big) = \dot{w}(w)$ as follows. The proof of Proposition 3.16 shows that $y := \dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet \in \mathbf{F}_p[\boldsymbol{r}_n(w)]^\bullet\langle \dot{w}(w)\rangle$ and $y \neq 0$, so $\dot{w}(w)$ is a weight of $\mathbf{F}_p[\boldsymbol{r}_n(w)]^\bullet$. By (11), there exist $N_0, \ldots, N_{l-1} \in \mathbb{N}$ such that

$$\dot{w}(w) = \dot{\mathrm{wt}}(|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet) - \sum_{i=0}^{l-1} N_i\, \dot{\alpha}_i. \tag{32}$$

Moreover, by definition of the Jacon elements and (11), the weight of $\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet$ with $\boldsymbol{\lambda}_n \in \Pi^n(\boldsymbol{v}_n(w), l)^\circ$ is equal to $\dot{\mathrm{wt}}(|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet) - \sum_{i=0}^{l-1} N_i(\boldsymbol{\lambda}_n; \boldsymbol{v}_n(w); l)\, \dot{\alpha}_i$. As a consequence, the multi-partitions $\boldsymbol{\lambda}_n$ we are looking for are the vertices of the crystal graph of $U_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{v}_n(w)\rangle^\bullet$ that have exactly N_i i-nodes for all $0 \le i \le l-1$. In a similar way, we compute the list of the multi-partitions $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l(w), n)^\circ$ satisfying the condition $\mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) = w'(w, \mu)$ ($\mu \in \Pi$, $|\mu| \le N(w)$).

We now give an example for Proposition 3.16.

Example 3.17 Take $n = 3$, $l = 2$, $\boldsymbol{s}_l = (3, 6)$ and $v = 2\Lambda_0 - 4\delta = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - (\alpha_0 + \alpha_1 + \alpha_2)$. We compute the basis $\mathbb{B}$ of $\mathbf{F}_q[\boldsymbol{s}_l]\langle v\rangle$ given by Proposition 3.16. By Example 3.15, the admissible weights are $2\Lambda_0 - 3\delta$, $(2\Lambda_0 - 3\delta) - \alpha_0$ and $(2\Lambda_0 - 3\delta) - (\alpha_0 + \alpha_1 + \alpha_2)$.

* Contribution of $w := 2\Lambda_0 - 3\delta$. Example 3.15 gives $\dot{w}(w) = \dot{\mathrm{wt}}(|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet) - 3\dot{\alpha}_1$, $\boldsymbol{v}_n(w) = (1, 1, 1)$, $\boldsymbol{r}_l(w) = (6, 3)$ and $\boldsymbol{v}_l(w) = (0, 0)$. By computing the first 4 layers of the crystal graph of $U_p(\widehat{\mathfrak{sl}}_l).|\boldsymbol{\emptyset}_n, \boldsymbol{v}_n(w)\rangle^\bullet$, we see that the only multi-partition $\boldsymbol{\lambda}_n$ in $\Pi^n(\boldsymbol{v}_n(w), l)^\circ$ such that $\dot{\mathrm{wt}}\big(\dot{F}(\boldsymbol{\lambda}_n).|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n(w)\rangle^\bullet\big) = \dot{w}(w)$ is $\boldsymbol{\lambda}_n := \big((1), (1), (1)\big)$. The corresponding Jacon element is
$$\dot{F}(\boldsymbol{\lambda}_n) = \dot{f}_1^{(3)}.$$
Moreover, since $N(w) = 1$, the only contributing partitions μ are $\mu = \emptyset$ and $\mu = (1)$. Take $\mu = \emptyset$. With notation from Proposition 3.16, we have
$$w'(w, \mu) = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle) - (\alpha_0 + \alpha_1 + \alpha_2),$$
so we compute the first 3 layers of the crystal graph of $U_q(\widehat{\mathfrak{sl}}_n).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle$. Doing this, we see that the only multi-partitions $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{v}_l(w), n)^\circ$ satisfying the condition $\mathrm{wt}\big(F(\boldsymbol{\lambda}_l).|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l(w)\rangle\big) = w'(w, \mu)$ are $\big(\emptyset, (3)\big)$ and $\big(\emptyset, (2, 1)\big)$. The corresponding Jacon elements are $f_2 f_1 f_0$ and $f_1 f_2 f_0$. We therefore get 2 vectors of $\mathbb{B}$, namely
$$v_1 := \dot{f}_1^{(3)} f_2 f_1 f_0.|\boldsymbol{\emptyset}_l, (6, 3)\rangle \qquad \text{and} \qquad v_2 := \dot{f}_1^{(3)} f_1 f_2 f_0.|\boldsymbol{\emptyset}_l, (6, 3)\rangle.$$
Taking $w = 2\Lambda_0 - 3\delta$, $\mu = (1)$ gives another vector of $\mathbb{B}$:
$$v_3 := \dot{f}_1^{(3)} B_{-1}.|\boldsymbol{\emptyset}_l, (6, 3)\rangle.$$

* Contribution of the other admissible weights. In the same way, we get 3 other vectors of $\mathbb{B}$:

$$v_4 := \dot{f}_1^{(3)}\dot{f}_0 f_2 f_1.|\boldsymbol{\emptyset}_l,(5,4)\rangle, \quad v_5 := \dot{f}_1^{(3)}\dot{f}_0 f_1 f_2.|\boldsymbol{\emptyset}_l,(5,4)\rangle \quad \text{and} \quad v_6 := \dot{f}_1^{(3)}\dot{f}_0\dot{f}_1.|\boldsymbol{\emptyset}_l,(6,3)\rangle.$$

By Proposition 3.16, we therefore have

$$\begin{aligned}\mathbb{B} &= \{v_1,\dots,v_6\} \\ &= \Big\{\dot{f}_1^{(3)} f_2 f_1 f_0.|\boldsymbol{\emptyset}_l,(6,3)\rangle, \quad \dot{f}_1^{(3)} f_1 f_2 f_0.|\boldsymbol{\emptyset}_l,(6,3)\rangle, \quad \dot{f}_1^{(3)} B_{-1}.|\boldsymbol{\emptyset}_l,(6,3)\rangle, \\ &\qquad \dot{f}_1^{(3)}\dot{f}_0 f_2 f_1.|\boldsymbol{\emptyset}_l,(5,4)\rangle, \quad \dot{f}_1^{(3)}\dot{f}_0 f_1 f_2.|\boldsymbol{\emptyset}_l,(5,4)\rangle, \quad \dot{f}_1^{(3)}\dot{f}_0\dot{f}_1.|\boldsymbol{\emptyset}_l,(6,3)\rangle\Big\}.\end{aligned}$$

$\diamond$

3.3 Action of the Heisenberg algebra $\mathcal{H}$ on the vectors $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$)

In this section, we explain how to compute the action of $\mathcal{H}$ on the vectors $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$) without using the straightening relations. Using the action of $U'_p(\widehat{\mathfrak{sl}}_l)$, we can restrict ourselves to the case where $\boldsymbol{s}_l = (s_1,\dots,s_l)$ is dominant, that is $s_1 \gg \cdots \gg s_l$ (see Definition 3.21). By a theorem of Uglov (see Theorem 3.22), this case can in turn be reduced to the level-one case. When $l = 1$, Leclerc and Thibon [LT2] have given a combinatorial formula for the action of $\mathcal{H}$, which avoids the straightening of q-wedge products in this case.

3.3.1 The case $l = 1$

We now recall the result of Leclerc and Thibon mentioned above. To this aim, we introduce the horizontal ribbon strips.

Definition 3.18 ([LT2], §4.1) Let θ be a skew diagram. Denote by $\theta{\downarrow}$ the horizontal strip made of the bottom nodes of the columns of θ. We say that θ is a *horizontal N-ribbon strip of weight k* if it can be tiled by k ribbons of length N whose origins lie in $\theta{\downarrow}$. One can check that if such a tiling exists, it is unique. In this case, the *spin* of θ, denoted by $\mathrm{spin}(\theta)$, is the sum of the heights of the ribbons in the tiling of θ. $\diamond$

Still following [LT2], we define operators of Λ^s as follows. Let λ, $\mu \in \Pi$ and N, $m \in \mathbb{N}^*$. If $\lambda \subset \mu$ and $\mu \setminus \lambda$ is a horizontal N-ribbon strip of weight m, put $L^{(N)}_{\lambda,\mu,m}(q) := (-q)^{-\mathrm{spin}(\theta)}$. Otherwise, put $L^{(N)}_{\lambda,\mu,m}(q) := 0$. Now define $\mathcal{U}_k$, $\mathcal{V}_k \in \mathrm{End}(\Lambda^s)$ ($k \in \mathbb{N}^*$) by

$$(33) \qquad \mathcal{U}_k.|\nu,s\rangle = \sum_{\mu\in\Pi} L^{(n)}_{\mu,\nu,k}(q)\,|\mu,s\rangle \quad \text{and} \quad \mathcal{V}_k.|\nu,s\rangle = \sum_{\lambda\in\Pi} L^{(n)}_{\nu,\lambda,k}(q)\,|\lambda,s\rangle \qquad (\nu\in\Pi).$$

Before stating the result of Leclerc and Thibon, let us give some extra notation. Let $\lambda \in \Pi$. Recall the notation B_λ and $B_{-\lambda}$ from (31), and define in a similar way $\mathcal{U}_\lambda$, $\mathcal{V}_\lambda \in \mathrm{End}(\Lambda^s)$. Let $\{h_\lambda \mid \lambda \in \Pi\}$ denote the set of complete symmetric functions and $\{p_\lambda \mid \lambda \in \Pi\}$ denote

the set of power sums [Mac]. Either set is a basis of the space of symmetric functions. Let $(\alpha_{\lambda,\mu})_{\lambda,\mu\in\Pi}$ denote the transition matrix between both bases, namely the matrix defined by

$$p_\mu = \sum_{\lambda\in\Pi} \alpha_{\lambda,\mu} h_\lambda \qquad (\mu \in \Pi). \tag{34}$$

The entries $\alpha_{\lambda,\mu}$ can be computed recursively, *e.g.* using Newton formula (see [Mac, (2.11)]):

$$p_{(k)} = k h_{(k)} - \sum_{i=1}^{k-1} p_{(i)} h_{(k-i)} \qquad (k \in \mathbb{N}^*). \tag{35}$$

Theorem 3.19 ([LT2], Thm. 6.4) *Assume that $l = 1$. Then with the notation above, we have*

$$B_k = \begin{cases} \sum_{\lambda\in\Pi} \alpha_{\lambda,(k)}\, \mathcal{U}_\lambda & \text{if} \quad k > 0, \\ \sum_{\lambda\in\Pi} \alpha_{\lambda,(|k|)}\, \mathcal{V}_\lambda & \text{if} \quad k < 0. \end{cases}$$

□

Example 3.20 Take $l = 1$, $n = 2$, $k = -2$ and $s \in \mathbb{Z}$. By Theorem 3.19 and (35), we have $B_{-2} = 2\mathcal{V}_2 - \mathcal{V}_1^2$. One computes

$$\begin{aligned}
\mathcal{V}_1.|\emptyset, s\rangle &= |(2), s\rangle - q^{-1}|(1,1), s\rangle, \\
\mathcal{V}_1.|(2), s\rangle &= |(4), s\rangle + |(2,2), s\rangle - q^{-1}|(2,1,1), s\rangle, \\
\mathcal{V}_1.|(1,1), s\rangle &= |(3,1), s\rangle - q^{-1}|(2,2), s\rangle - q^{-1}|(1,1,1,1), s\rangle, \\
\mathcal{V}_1^2.|\emptyset, s\rangle &= |(4), s\rangle - q^{-1}|(3,1), s\rangle + (1+q^{-2})|(2,2), s\rangle - q^{-1}|(2,1,1), s\rangle + q^{-2}|(1,1,1,1), s\rangle, \\
\mathcal{V}_2.|\emptyset, s\rangle &= |(4), s\rangle - q^{-1}|(3,1), s\rangle + q^{-2}|(2,2), s\rangle,
\end{aligned}$$

whence

$$B_{-2}.|\emptyset, s\rangle = |(4), s\rangle - q^{-1}|(3,1), s\rangle + (-1+q^{-2})|(2,2), s\rangle + q^{-1}|(2,1,1), s\rangle - q^{-2}|(1,1,1,1), s\rangle,$$

as one can check using the straightening relations. ◇

3.3.2 The dominant case

We shall now see that by a result of [U] (see Theorem 3.22), it is possible to compute the action of B_m ($m \in \mathbb{Z}^*$) on certain vectors of the standard basis of Λ^s by reducing to the case $l = 1$. In this case, we can apply subsequently Theorem 3.19 and therefore avoid using the straightening relations. Before stating Theorem 3.22, let us introduce the following definition and notation.

Definition 3.21 ([U]) Let $M \in \mathbb{N}^*$, $\boldsymbol{\lambda}_l \in \Pi^l$ and $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$. We say that the pair $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is *M-dominant* if for all $1 \leq i \leq l-1$, we have

$$s_i - s_{i+1} \geq M + |\boldsymbol{\lambda}_l|.$$

$\diamond$

Recall that $\Lambda^s[n,1]$ denotes the space of q-wedge products of charge s in which we take $l=1$. If x is an operator of $\Lambda^s[n,1]$ that acts on the standard basis by

$$x.|\lambda, s\rangle = \sum_{\mu \in \Pi} x^{(s)}_{\lambda,\mu} \, |\mu, s\rangle \qquad \big(\lambda \in \Pi,\, x^{(s)}_{\lambda,\mu} \in \mathbb{Q}(q)\big), \tag{36}$$

define for $b \in [\![1;l]\!]$ an operator $x[b]$ of $\Lambda^s[n,l]$ by

$$\begin{gathered} x[b].|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = \sum_{\mu \in \Pi} x^{(s_b)}_{\lambda^{(b)},\mu} \left|(\lambda^{(1)}, \dots, \lambda^{(b-1)}, \mu, \lambda^{(b+1)}, \dots, \lambda^{(l)}), \boldsymbol{s}_l\right\rangle \\ (\boldsymbol{\lambda}_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi^l, \quad \boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l(s))\,; \end{gathered} \tag{37}$$

in other words, $x[b]$ acts as x on the b-th component and trivially on the other components.

Theorem 3.22 ([U], Prop 5.3) *Let $m \in \mathbb{Z}^*$. Assume that $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is $n\widetilde{m}$-dominant, where $\widetilde{m} := \max(0, -m)$. Then we have*

$$B_m.|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = \sum_{b=1}^{l} q^{(b-1)|m|} \, B_m[b].|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle.$$

$\square$

Example 3.23 Take $n=2$, $l=2$, $m=-2$, $\boldsymbol{\lambda}_l = \boldsymbol{\emptyset}_l$ and $\boldsymbol{s}_l = (2,-2)$. By Theorem 3.22 and Example 3.20, we have

$$\begin{aligned} B_{-2}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle &= B_{-2}[1].|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle + q^2 B_{-2}[2].|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle \\ &= |((4),\emptyset), \boldsymbol{s}_l\rangle - q^{-1}|((3,1),\emptyset), \boldsymbol{s}_l\rangle + (-1+q^{-2})|((2,2),\emptyset), \boldsymbol{s}_l\rangle \\ &\quad + q^{-1}|((2,1,1),\emptyset), \boldsymbol{s}_l\rangle - q^{-2}|((1,1,1,1),\emptyset), \boldsymbol{s}_l\rangle \\ &\quad + q^2|(\emptyset,(4)), \boldsymbol{s}_l\rangle - q|(\emptyset,(3,1)), \boldsymbol{s}_l\rangle + (-q^2+1)|(\emptyset,(2,2)), \boldsymbol{s}_l\rangle \\ &\quad + q|(\emptyset,(2,1,1)), \boldsymbol{s}_l\rangle - |(\emptyset,(1,1,1,1)), \boldsymbol{s}_l\rangle, \end{aligned}$$

as one can check using the straightening relations. $\diamond$

3.3.3 Action of $\mathcal{H}$ on $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$, $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$

Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \Pi$ and $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Recall that we put $B_{-\lambda} := B_{-\lambda_1} \cdots B_{-\lambda_r} \in \mathcal{H}$. We derive here a method for computing $B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$ without using the straightening relations. We proceed in four steps. The first two steps use some results that we prove in this section.

1. We find $\boldsymbol{t}_l \in \dot{W}_l.\boldsymbol{s}_l$ such that $\boldsymbol{t}_l$ is $n|\lambda|$-dominant; see Lemma 3.24.
2. We find $\dot{u} \in U'_p(\widehat{\mathfrak{sl}}_l)$ (more precisely, $\dot{u}$ is a monomial in the Chevalley generators of $U'_p(\widehat{\mathfrak{sl}}_l)$) such that $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle = \dot{u}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$; see Proposition 3.26.
3. We claim that we can compute $B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$ by repeated applications of Theorem 3.22. Assume that $r = 2$, *i.e.* $\lambda = (\lambda_1, \lambda_2)$ has only 2 parts (the general case follows by induction on r). By Theorem 3.22, $B_{-\lambda_1}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$ is a linear combination of vectors of the form $|\boldsymbol{\mu}_l, \boldsymbol{t}_l\rangle$ with $\boldsymbol{\mu}_l \in \Pi^l$. Let $\boldsymbol{\mu}_l \in \Pi^l$ be such that $|\boldsymbol{\mu}_l, \boldsymbol{t}_l\rangle$ appears in this linear combination. Put $w := \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle)$. Since $B_{-\lambda_1}.\Lambda^s\langle w\rangle \subset \Lambda^s\langle w - \lambda_1\delta\rangle$, Equation (9) implies that $|\boldsymbol{\mu}_l| = n\lambda_1$, so $(\boldsymbol{\mu}_l, \boldsymbol{t}_l)$ is $n\lambda_2$-dominant. We can therefore apply Theorem 3.22 to compute $B_{-\lambda_2}.|\boldsymbol{\mu}_l, \boldsymbol{t}_l\rangle$, which proves the claim.
4. Since the actions of $U'_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ commute, we have $B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle = \dot{u}.(B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle)$. Finally, we compute the action of $\dot{u}$ on $B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$ by using the indexation of the standard basis of Λ^s by n-multi-partitions.

We now state and prove Lemma 3.24 and Proposition 3.26. We give an example of application at the end of this section.

Lemma 3.24 *Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ and $M \in \mathbb{N}^*$. Then there exists an nM-dominant multi-charge $\boldsymbol{t}_l$ that is $\dot{W}_l$-conjugated to $\boldsymbol{s}_l$.*

Proof. Let $\lambda := ((l-1)M, (l-2)M, \dots, M, -l(l-1)M/2) \in \mathbb{Z}^l$ and $\dot{\tau} \in \mathrm{End}(\mathbb{Q}^l)$ be the translation by $n\lambda$. Since $\lambda \in \mathbb{Z}^l(0)$, we have $\dot{\tau} \in \dot{W}_l$. We can choose $\dot{\sigma} \in \mathfrak{S}_l \subset \dot{W}_l$ such that $\boldsymbol{a}_l = (a_1, \dots, a_l) := \dot{\sigma}.\boldsymbol{s}_l$ satisfies $a_1 \geq a_2 \geq \dots \geq a_l$. Then $\boldsymbol{t}_l := \dot{\tau}\dot{\sigma}.\boldsymbol{s}_l$ is nM-dominant and $\dot{W}_l$-conjugated to $\boldsymbol{s}_l$. □

Recall the definition of $\sigma_i(\boldsymbol{\lambda}_l) \in \Pi^l$ ($\boldsymbol{\lambda}_l \in \Pi^l$, $0 \leq i \leq n-1$) from Definition 2.8. In a similar way, one defines for $0 \leq i \leq l-1$ and $\boldsymbol{\lambda}_n \in \Pi^n$ an n-multi-partition $\dot{\sigma}_i(\boldsymbol{\lambda}_n)$.

Lemma 3.25 *Let $\boldsymbol{r}_l \in A_{l,n}(s)$. Let $\boldsymbol{r}_n$ be the unique multi-charge in $A_{n,l}(s)$ such that $|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = |\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$. Let $\dot{\sigma} = \dot{\sigma}_{i_r} \dots \dot{\sigma}_{i_1}$ be a reduced expression of $\dot{\sigma} \in \dot{W}_l$. Define a sequence $(\boldsymbol{\lambda}_n^{(0)}, \dots, \boldsymbol{\lambda}_n^{(r)})$ of n-multi-partitions by*

$$\boldsymbol{\lambda}_n^{(0)} := \boldsymbol{\emptyset}_n \qquad \text{and} \qquad \boldsymbol{\lambda}_n^{(j)} := \dot{\sigma}_{i_j}(\boldsymbol{\lambda}_n^{(j-1)}) \quad (1 \leq j \leq r).$$

Then for all $1 \leq j \leq r$, $\dot{\mathrm{wt}}\big(|\boldsymbol{\lambda}_n^{(j-1)}, \boldsymbol{r}_n\rangle^\bullet\big) + \dot{\alpha}_{i_j}$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$, and we have

$$|\boldsymbol{\lambda}_n^{(r)}, \boldsymbol{r}_n\rangle^\bullet = |\boldsymbol{\emptyset}_l, \dot{\sigma}.\boldsymbol{r}_l\rangle.$$

Proof. We proceed by induction on r. If $r=0$, the equality $|\boldsymbol{\lambda}_n^{(r)},\boldsymbol{r}_n\rangle^\bullet = |\boldsymbol{\emptyset}_l,\dot{\sigma}.\boldsymbol{r}_l\rangle$ is obvious and (11) in Prop. 2.1 shows that for all $i_1\in[\![0;l-1]\!]$, $\dot{\mathrm{wt}}\big(|\boldsymbol{\lambda}_n^{(0)},\boldsymbol{r}_n\rangle^\bullet\big)+\dot{\alpha}_{i_1}$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$. Assume now that $r>0$. Put $\dot{\tau}:=\dot{\sigma}_{i_{r-1}}\cdots\dot{\sigma}_{i_1}$, $\boldsymbol{s}_l=(s_1,\dots,s_l):=\dot{\sigma}.\boldsymbol{r}_l$, $\boldsymbol{t}_l=(t_1,\dots,t_l):=\dot{\tau}.\boldsymbol{r}_l$, $u:=|\boldsymbol{\lambda}_n^{(r-1)},\boldsymbol{r}_n\rangle^\bullet$, $\dot{w}:=\dot{\mathrm{wt}}(u)$, $x:=|\boldsymbol{\emptyset}_l,\boldsymbol{s}_l\rangle$, $y:=|\boldsymbol{\lambda}_n^{(r)},\boldsymbol{r}_n\rangle^\bullet$ and $i:=i_r$. By induction, we have $u=|\boldsymbol{\emptyset}_l,\boldsymbol{t}_l\rangle$. We must show that $x=y$ and $\dot{w}+\dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$.
We claim that x and y lie in $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot{\sigma}_i.\dot{w}\rangle$. The claim for y follows by the definition of $\boldsymbol{\lambda}_n^{(r)}=\dot{\sigma}_i(\boldsymbol{\lambda}_n^{(r-1)})\in\Pi^n$. Since $\boldsymbol{s}_l\in\dot{W}_l.\boldsymbol{t}_l$, we have $\Lambda_{s_1}+\cdots+\Lambda_{s_l}=\Lambda_{t_1}+\cdots+\Lambda_{t_l}$. Therefore (9) implies $\mathrm{wt}(x)\equiv\mathrm{wt}(u) \bmod \mathbb{Z}\delta$. Moreover, Proposition 2.4 implies that $u=|\boldsymbol{\lambda}_n^{(r-1)},\boldsymbol{r}_n\rangle^\bullet$ is in $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet=\displaystyle\bigoplus_{d\in\mathbb{Z}}\Lambda^s\langle a_0\Lambda_0+\cdots+a_{n-1}\Lambda_{n-1}+d\delta\rangle$ with $(a_0,\dots,a_{n-1}):=\theta_{n,l}(\boldsymbol{r}_n)$. This implies that $x\in\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$. Now we show easily that $\dot{\mathrm{wt}}(x)=\dot{\sigma}_i.\dot{\mathrm{wt}}(u)=\dot{\mathrm{wt}}(y)$, whence the claim for x. Put $\dot{w}_\emptyset:=\dot{\mathrm{wt}}\big(|\boldsymbol{\emptyset}_n,\boldsymbol{r}_n\rangle^\bullet\big)$. By (11), the weight subspace $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot{w}_\emptyset\rangle$ is one-dimensional. Since the formal character of the integrable $U_p(\widehat{\mathfrak{sl}}_l)$-module $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$ is $\dot{W}_l$-invariant, we have

$$\dim(\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot{\sigma}_i.\dot{w}\rangle)=\dim(\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot{\sigma}.\dot{w}_\emptyset\rangle)=\dim(\mathbf{F}_p[\boldsymbol{r}_n]^\bullet\langle\dot{w}_\emptyset\rangle)=1.$$

As a consequence, x and y are vectors of the standard basis of a one-dimensional subspace of Λ^s, whence $x=y$.

Let us now show that $\dot{w}+\dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$. Put

$$\dot{\varepsilon}_i:=\max\{m\in\mathbb{N}\mid \dot{e}_i^m.x\neq 0\}\qquad\text{and}\qquad \dot{\varphi}_i:=\max\{m\in\mathbb{N}\mid \dot{f}_i^m.x\neq 0\}.$$

The theory of $U_p(\mathfrak{sl}_2)$-modules yields

$$(\dot{\mathrm{wt}}(x),\,\dot{\alpha}_i)=\dot{\varphi}_i-\dot{\varepsilon}_i.$$

By (10), we have $(\dot{\mathrm{wt}}(x),\,\dot{\alpha}_i)=s_i-s_{i+1}$, where we put $s_0:=s_l+n$ if $i=0$. We thus have $\dot{\varphi}_i-\dot{\varepsilon}_i=s_i-s_{i+1}$. Since $\ell(\dot{\sigma})=\ell(\dot{\tau})+1$, a classical result about Coxeter groups (see *e.g.* [D, Lemma 2.1 (iii)]) shows that $s_{i+1}>s_i$ (even if $i=0$). As a consequence, $\dot{\varepsilon}_i>\dot{\varphi}_i\geq 0$, therefore $\dot{\mathrm{wt}}(x)+\dot{\alpha}_i$ and $\dot{\sigma}^{-1}.(\dot{\mathrm{wt}}(x)+\dot{\alpha}_i)=\dot{w}_\emptyset+\dot{\sigma}^{-1}.\dot{\alpha}_i$ are weights of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$. By (11), we have $\dot{\sigma}^{-1}.\dot{\alpha}_i\in\sum_{j=0}^{l-1}\mathbb{N}\dot{\alpha}_j$ and $\dot{w}_\emptyset-\dot{\sigma}^{-1}.\dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$. Therefore $\dot{w}+\dot{\alpha}_i=\dot{\tau}.(\dot{w}_\emptyset-\dot{\sigma}^{-1}.\dot{\alpha}_i)$ is not a weight of $\mathbf{F}_p[\boldsymbol{r}_n]^\bullet$ either. □

The following result is a refinement of [U, Cor. 4.9].

Proposition 3.26 *Let $\boldsymbol{r}_l\in A_{l,n}(s)$ and $\boldsymbol{s}_l=(s_1,\dots,s_l)\in\dot{W}_l.\boldsymbol{r}_l$. Then there exist integers $k_1,\dots,k_r\in\mathbb{N}$, $i_1,\dots,i_r\in[\![0;l-1]\!]$ (which can be explicitly calculated) such that*

$$|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l\rangle=\dot{e}_{i_1}^{(k_1)}\cdots\dot{e}_{i_r}^{(k_r)}.|\boldsymbol{\emptyset}_l,\boldsymbol{s}_l\rangle\qquad\textit{and}\qquad|\boldsymbol{\emptyset}_l,\boldsymbol{s}_l\rangle=\dot{f}_{i_r}^{(k_r)}\cdots\dot{f}_{i_1}^{(k_1)}.|\boldsymbol{\emptyset}_l,\boldsymbol{r}_l\rangle.$$

Proof. Let $\dot{\sigma}\in\dot{W}_l$ be the element of minimal length such that $\boldsymbol{s}_l=(s_1,\dots,s_l)=\dot{\sigma}.\boldsymbol{r}_l$. We argue by induction on the length r of $\dot{\sigma}$. If $r=0$, we have $\boldsymbol{s}_l=\boldsymbol{r}_l$ and there is nothing to prove. Assume now that $r>0$. We compute a reduced expression

$$\dot{\sigma}:=\dot{\sigma}_{i_r}\cdots\dot{\sigma}_{i_1}$$

of $\dot{\sigma}$ as follows. Since $r > 0$, we have $\boldsymbol{s}_l \notin A_{l,n}(s)$, so by (14) there exists $i_r \in [\![0; l-1]\!]$ such that $s_{i_r} < s_{i_r+1}$ (here we put, as usual, $s_0 := n + s_l$). Let $\dot{\tau} := \dot{\sigma}_{i_r}\dot{\sigma}$. By [D, Lemma 2.1 (iii)], we have $\ell(\dot{\tau}) < \ell(\dot{\sigma})$. By induction we get a reduced expression $\dot{\tau} = \dot{\sigma}_{i_{r-1}} \cdots \dot{\sigma}_{i_1}$ of $\dot{\tau}$, so $\dot{\sigma} = \dot{\sigma}_{i_r} \cdots \dot{\sigma}_{i_1}$ is a reduced expression of $\dot{\sigma}$. Now let $\boldsymbol{r}_n \in A_{n,l}(s)$ be such that $|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^{\bullet} = |\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$. By Lemma 3.25, we have $|\boldsymbol{\emptyset}_l,\, \boldsymbol{s}_l\rangle = |\dot{\sigma}_{i_r} \ldots \dot{\sigma}_{i_1}(\boldsymbol{\emptyset}_n), \boldsymbol{r}_n\rangle^{\bullet}$. Let

$$k_j := |\dot{\sigma}_{i_j} \ldots \dot{\sigma}_{i_1}(\boldsymbol{\emptyset}_n)| - |\dot{\sigma}_{i_{j-1}} \ldots \dot{\sigma}_{i_1}(\boldsymbol{\emptyset}_n)| \qquad (1 \leq j \leq r).$$

By Proposition 2.9, we have

$$|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = |\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^{\bullet} = \dot{e}_{i_1}^{(k_1)} \cdots \dot{e}_{i_r}^{(k_r)}.|\dot{\sigma}_{i_r} \ldots \dot{\sigma}_{i_1}(\boldsymbol{\emptyset}_n), \boldsymbol{r}_n\rangle^{\bullet} = \dot{e}_{i_1}^{(k_1)} \cdots \dot{e}_{i_r}^{(k_r)}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle,$$

and the proof of the other equality is similar. □

Example 3.27 Take $s = 0$, $n = 2$, $l = 3$, $\lambda = (1)$ and $\boldsymbol{s}_l = (1, -1, 0)$. We apply the method derived in this section for computing $B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle$. Note that $\boldsymbol{s}_l$ is not $n|\lambda|$-dominant, so we cannot directly apply Theorem 3.22. The computation is done in four steps.

1. Following the proof of Lemma 3.24, put $\boldsymbol{t}_l := (3, 1, -4)$. Then $\boldsymbol{t}_l$ is $n|\lambda|$-dominant and $\dot{W}_l$-conjugated to $\boldsymbol{s}_l$.

2. As in the proof of Proposition 3.26, we compute $\dot{u} \in U_p'(\widehat{\mathfrak{sl}}_l)$ such that $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle = \dot{u}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$. The element in $A_{l,n}(s)$ that is $\dot{W}_l$-conjugated to $\boldsymbol{s}_l$ is $\boldsymbol{r}_l := (1, 0, -1)$. The following array shows the computation of a reduced expression of $\dot{\sigma}$, where $\dot{\sigma} \in \dot{W}_l$ is the element of minimal length such that $\boldsymbol{s}_l = \dot{\sigma}.\boldsymbol{r}_l$. Each line contains a multi-charge $\boldsymbol{a}_l = (a_1, \ldots, a_l)$ and an integer $i \in [\![0; l-1]\!]$ such that $a_i < a_{i+1}$ (if it exists). The multi-charge on the next line is $\dot{\sigma}_i.\boldsymbol{a}_l$.

$$\begin{array}{ll} (3,1,-4) & i=0 \\ (-2,1,1) & i=1 \\ (1,-2,1) & i=2 \\ (1,1,-2) & i=0 \\ (0,1,-1) & i=1 \\ (1,0,-1) & \end{array}$$

We get $\boldsymbol{s}_l = \dot{\sigma}_{i_r} \cdots \dot{\sigma}_{i_1}.\boldsymbol{r}_l$ with $r = 5$ and $(i_r, \ldots, i_1) = (0, 1, 2, 0, 1)$. The element $\boldsymbol{r}_n \in A_{n,l}(s)$ such that $|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = |\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^{\bullet}$ is $\boldsymbol{r}_n := (1, -1)$. We compute recursively the n-multi-partitions $\boldsymbol{\lambda}_n^{(j)} := \dot{\sigma}_{i_j} \cdots \dot{\sigma}_{i_1}(\boldsymbol{\emptyset}_n)$ $(0 \leq j \leq r)$ from $\boldsymbol{\lambda}_n^{(0)} = \boldsymbol{\emptyset}_n$. The proof of Proposition 3.26 shows that for $1 \leq j \leq r$, k_j is the number of addable i_j-nodes of $\boldsymbol{\lambda}_n^{(j-1)}$ and $\boldsymbol{\lambda}_n^{(j)}$ is obtained from $\boldsymbol{\lambda}_n^{(j-1)}$ by adding these nodes. The computation of the integers k_j $(1 \leq j \leq r)$ is displayed in the following array.

j	$\boldsymbol{\lambda}_n^{(j)}$	i_{j+1}	k_{j+1}
0	$((),())$	1	1
1	$((1),())$	0	1
2	$((1,1),())$	2	3
3	$((2,1,1),(1))$	1	3
4	$((2,2,1,1),(1,1))$	0	5
5	$((3,2,2,1,1),(2,1,1))$		

By Proposition 3.26, we therefore have

$$|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = \dot{e}_1^{(1)}\dot{e}_0^{(1)}\dot{e}_2^{(3)}\dot{e}_1^{(3)}\dot{e}_0^{(5)}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle.$$

In the same way we get

$$|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle = \dot{f}_2^{(1)}.|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle.$$

As a consequence, we have $|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle = \dot{u}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle$ with

$$\dot{u} = \dot{f}_2^{(1)}\dot{e}_1^{(1)}\dot{e}_0^{(1)}\dot{e}_2^{(3)}\dot{e}_1^{(3)}\dot{e}_0^{(5)}.$$

3. By Theorem 3.22, we have

$$\begin{aligned} B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle &= |((2),\emptyset,\emptyset),\boldsymbol{t}_l\rangle - q^{-1}|((1,1),\emptyset,\emptyset),\boldsymbol{t}_l\rangle \\ &\quad + q|(\emptyset,(2),\emptyset),\boldsymbol{t}_l\rangle - |(\emptyset,(1,1),\emptyset),\boldsymbol{t}_l\rangle \\ &\quad + q^2|(\emptyset,\emptyset,(2)),\boldsymbol{t}_l\rangle - q|(\emptyset,\emptyset,(1,1)),\boldsymbol{t}_l\rangle. \end{aligned}$$

4. We only have to apply $\dot{u}$ to get

$$\begin{aligned} B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle &= \dot{u}.(B_{-\lambda}.|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle) \\ &= |((2),\emptyset,\emptyset),\boldsymbol{s}_l\rangle + |(\emptyset,\emptyset,(2)),\boldsymbol{s}_l\rangle + (q-q^{-1})|((1),\emptyset,(1)),\boldsymbol{s}_l\rangle \\ &\quad - |(\emptyset,\emptyset,(1,1)),\boldsymbol{s}_l\rangle + q^2|(\emptyset,(2),\emptyset),\boldsymbol{s}_l\rangle - |((1,1),\emptyset,\emptyset),\boldsymbol{s}_l\rangle \\ &\quad - q|(\emptyset,(1,1),\emptyset),\boldsymbol{s}_l\rangle, \end{aligned}$$

as one can check using the straightening relations. $\diamond$

3.4 End of the algorithm

Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ and w be a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. By Sections 3.2 and 3.3, we get a basis $\mathbb{B}$ of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ that is $^-$ -invariant. Let $T(q)$ denote the transition matrix between the standard basis of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ and the basis $\mathbb{B}$, and let $A(q)$ denote the matrix of the involution $^-$ of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ with respect to the standard basis. Since the basis $\mathbb{B}$ is $^-$ -invariant, we have the following:

Lemma 3.28 *With the notation above, we have $A(q) = T(q)\big(T(q^{-1})\big)^{-1}$.* $\square$

Thus this lemma gives an algorithm for computing the involution $^-$ of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$. Once this is known, the computation of the canonical bases can be performed in a classical way by solving unitriangular systems (see *e.g.* [L, Proof of Thm. 7.1]).

4 Examples

In this section, we give some transition matrices of the canonical bases of some weight subspaces of higher-level Fock spaces. We first give, for $k \in \mathbb{Z}$, the matrices

$$\Delta_k(q) = \big(\Delta^+_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l;\,\boldsymbol{s}_l^{(k)}}(q)\big)_{\boldsymbol{\lambda}_l,\,\boldsymbol{\mu}_l \in \Pi^l(\boldsymbol{s}_l^{(k)};\,w^{(k)})}$$

for $n = l = 2$, $\boldsymbol{s}_l^{(k)} = (2k, -2k)$ and $w^{(k)} = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l^{(k)}\rangle) - (2\alpha_0 + 2\alpha_1)$. These matrices should be read by columns, for example we have

$$G^+\Big(\big((3,1),\emptyset\big),(-2,2)\Big) = \Big|\big((3,1),\emptyset\big),(-2,2)\Big\rangle + q\,\Big|\big((2,2),\emptyset\big),(-2,2)\Big\rangle + q^2\,\Big|\big((2,1,1),\emptyset\big),(-2,2)\Big\rangle.$$

Let N_k denote the number of nonzero entries in $\Delta_k(q)$. Mark by a $*$ the rows indexed by the multi-partitions in $\Pi^l(\boldsymbol{s}_l^{(k)}, n)^\circ$. As a consequence, the corresponding columns give the expression (in the standard basis) of the vectors of the lower global crystal basis of the irreducible $U_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{M}_q[\boldsymbol{s}_l^{(k)}]$.

Using Uglov's algorithm, we were able to compute the matrices $\Delta_k(q)$ only for $|k| \leq 1$, because otherwise the number of factors of the q-wedge products to be straightened was too large. With our algorithm we computed the matrices $\Delta_k(q)$ for $|k| \lessapprox 20$. The matrix $\Delta_0(q)$ was already published in [U]. Note that for a given $k \geq 1$, the matrices $\Delta_k(q)$ and $\Delta_{-k}(q)$ are equal up to a permutation of the rows and the columns; if $k \geq 2$, this is a special case of [Y2, Prop. 5.5]. Moreover, note that for all $k \geq 1$ (case of the dominant multi-charges), the matrices $\Delta_k(q)$ are equal (up to a permutation of the rows and the columns). This property was observed and proved (for $k \geq 6$) in [Y2, Thm 5.2]. This result supports in turn a conjecture for computing the q-decomposition matrices of some cyclotomic v-Schur algebras (see Section 1 and [Y1]). In this conjecture, the assumption of dominance is necessary. Indeed, note that in our example we have $N_k \neq N_0$ for $k \neq 0$, which shows that $\Delta_k(1)$ $(k \neq 0)$ cannot be obtained from $\Delta_0(1)$ by a mere permutation of rows and columns.

$*$ The matrices $\Delta_k(q)$ for $k \leq -1$

$$\Delta_k(q) = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & q & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & q^2 & q & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
q^2 & q^3 & q^2 & q & 1 & . & . & . & . & . & . & . & . & . & . & . \\
q & 0 & 0 & 0 & q & 1 & . & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & q^2 & q & 1 & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & 0 & 0 & 0 & 1 & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & q^2 & q & 0 & 0 & 1 & . & . & . & . & . & . & . \\
q^3 & 0 & q & 2q^2 & q^3 & q^2 & q & q & q & 1 & . & . & . & . & . & . \\
q^2 & q & q^2 & q^3 & q^4 & q^3+q & q^2 & 0 & q^2 & q & 1 & . & . & . & . & . \\
q^2 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 1 & . & . & . & . \\
q^3 & q^2 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & q & q & 1 & . & . & . \\
0 & q^3 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & q & q^2 & 0 & q & 1 & . & . \\
q^3 & q^4 & q^3 & q^2 & 0 & q^2 & q & 0 & q & q^2 & q^3 & q & q^2 & q & 1 & . \\
q^4 & 0 & 0 & q^3 & 0 & q^3 & q^2 & q^2 & q^2 & q & 0 & q^2 & 0 & 0 & q & 1
\end{pmatrix}
\begin{array}{l}
*\,\big(\emptyset,(4)\big) \\
*\,\big(\emptyset,(3,1)\big) \\
\big(\emptyset,(2,2)\big) \\
\big(\emptyset,(2,1,1)\big) \\
*\,\big((1),(2,1)\big) \\
\big((2),(2)\big) \\
\big((1,1),(2)\big) \\
\big(\emptyset,(1,1,1,1)\big) \\
\big((2),(1,1)\big) \\
\big((1,1),(1,1)\big) \\
\big((2,1),(1)\big) \\
\big((4),\emptyset\big) \\
\big((3,1),\emptyset\big) \\
\big((2,2),\emptyset\big) \\
\big((2,1,1),\emptyset\big) \\
\big((1,1,1,1),\emptyset\big)
\end{array},$$

$N_k = 87$

$*$ <u>The matrix $\Delta_0(q)$</u>

$$
\Delta_0(q) = \begin{array}{cl}
\left(\begin{array}{cccccccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 0 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q^2 & q & q & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & q & 0 & 0 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & q & 0 & 0 & 1 & . & . & . & . & . & . & . & . & . & . \\
q & q^2 & 0 & 0 & q & q & 1 & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & q & 0 & 0 & q^2 & q & 1 & . & . & . & . & . & . & . & . \\
q^2 & q^3 & 0 & 0 & q^2 & 0 & q & 0 & 1 & . & . & . & . & . & . & . \\
q^2 & 0 & q & 0 & 0 & q^2 & q & 0 & 0 & 1 & . & . & . & . & . & . \\
q^3 & q^2 & 2q^2 & q & q & q^3 & q^2 & q & 0 & q & 1 & . & . & . & . & . \\
q^2 & 0 & q^3 & 0 & q^2 & q^4 & q^3+q & q^2 & 0 & q^2 & q & 1 & . & . & . & . \\
q^3 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & q & 0 & 0 & q & 1 & . & . & . \\
0 & q^3 & 0 & q^2 & q^2 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 1 & . & . \\
q^3 & q^4 & q^2 & q^3 & q^3 & 0 & q^2 & q & q & q & q^2 & 0 & 0 & q & 1 & . \\
q^4 & 0 & q^3 & 0 & 0 & 0 & q^3 & q^2 & q^2 & q^2 & q & q^2 & q & 0 & q & 1
\end{array}\right) &
\begin{array}{l}
*\,(\emptyset,(4)) \\
*\,(\emptyset,(3,1)) \\
((4),\emptyset) \\
((3,1),\emptyset) \\
(\emptyset,(2,2)) \\
*\,((2),(2)) \\
((1),(2,1)) \\
((1,1),(2)) \\
(\emptyset,(2,1,1)) \\
((2),(1,1)) \\
((2,1),(1)) \\
((1,1),(1,1)) \\
(\emptyset,(1,1,1,1)) \\
((2,2),\emptyset) \\
((2,1,1),\emptyset) \\
((1,1,1,1),\emptyset)
\end{array}
\end{array},
$$

$N_0 = 86$

$*$ <u>The matrices $\Delta_k(q)$ for $k \geq 1$</u>

$$
\Delta_k(q) = \begin{array}{cl}
\left(\begin{array}{cccccccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & q & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & q^2 & q & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
q^2 & q^3 & q^2 & q & 1 & . & . & . & . & . & . & . & . & . & . & . \\
q & 0 & 0 & 0 & q & 1 & . & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & q^2 & q & 1 & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & 0 & 0 & 0 & 1 & . & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & q^2 & q & 0 & 0 & 1 & . & . & . & . & . & . & . \\
q^3 & 0 & q & 2q^2 & q^3 & q^2 & q & q & q & 1 & . & . & . & . & . & . \\
q^2 & q & q^2 & q^3 & q^4 & q^3+q & q^2 & 0 & q^2 & q & 1 & . & . & . & . & . \\
q^2 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 1 & . & . & . & . \\
q^3 & q^2 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & q & q & 1 & . & . & . \\
0 & q^3 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & q & q^2 & 0 & q & 1 & . & . \\
q^3 & q^4 & q^3 & q^2 & 0 & q^2 & q & 0 & q & q^2 & q^3 & q & q^2 & q & 1 & . \\
q^4 & 0 & 0 & q^3 & 0 & q^3 & q^2 & q^2 & q^2 & q & 0 & q^2 & 0 & 0 & q & 1
\end{array}\right) &
\begin{array}{l}
*\,((4),\emptyset) \\
*\,((3,1),\emptyset) \\
((2,2),\emptyset) \\
((2,1,1),\emptyset) \\
*\,((2,1),(1)) \\
((2),(2)) \\
((2),(1,1)) \\
((1,1,1,1),\emptyset) \\
((1,1),(2)) \\
((1,1),(1,1)) \\
((1),(2,1)) \\
(\emptyset,(4)) \\
(\emptyset,(3,1)) \\
(\emptyset,(2,2)) \\
(\emptyset,(2,1,1)) \\
(\emptyset,(1,1,1,1))
\end{array}
\end{array},
$$

$N_k = 87$

We now give the matrix $\Delta(q) = \big(\Delta^+_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l;\,\boldsymbol{s}_l}(q)\big)_{\boldsymbol{\lambda}_l,\,\boldsymbol{\mu}_l\in\Pi^l(\boldsymbol{s}_l;\,w)}$ for $n = l = 2$, $\boldsymbol{s}_l = (0,0)$ and $w = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - (7\alpha_0 + 4\alpha_1)$. It would be very hard to compute it using Uglov's algorithm. Indeed, the partitions λ such that $|\lambda, 0\rangle \in \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ all satisfy $|\lambda| = 25$, so applying Uglov's algorithm requires the straightening of q-wedge products with at least 25 factors. Moreover, since $\dim(\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle) = 28$, the matrix $\Delta(q)$ here is too large to be displayed on a single page. We thus write it as

$$\Delta(q) = \begin{pmatrix} \Delta^{(1,1)}(q) & 0 \\ \Delta^{(2,1)}(q) & \Delta^{(2,2)}(q) \end{pmatrix},$$

where $\Delta^{(1,1)}(q)$, $\Delta^{(2,1)}(q)$ and $\Delta^{(2,2)}(q)$ are (14×14)-matrices.

$$
\Delta^{(1,1)}(q) = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
q & 0 & q & 1 & . & . & . & . & . & . & . & . & . & . \\
q^2 & q & 0 & q & 1 & . & . & . & . & . & . & . & . & . \\
q & 0 & q & q^2 & q & 1 & . & . & . & . & . & . & . & . \\
q^2 & 0 & q^2 & 0 & 0 & q & 1 & . & . & . & . & . & . & . \\
q^2 & 0 & 0 & q & 0 & 0 & 0 & 1 & . & . & . & . & . & . \\
q^3 & q^2 & q & q^2 & q & 0 & 0 & q & 1 & . & . & . & . & . \\
0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & q & 1 & . & . & . & . \\
q^2 & q & q^2 & q^3 & q^2 & q & 0 & q^2 & q & 0 & 1 & . & . & . \\
q^3 & q^2 & q^3 & 0 & q & q^2 & q & 0 & q^2 & q & q & 1 & . & . \\
0 & 0 & q^2 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & . \\
0 & 0 & q^3 & q^2 & q & 0 & 0 & q & q^2 & q & 0 & 0 & q & 1
\end{pmatrix}
\begin{array}{l}
*\big((1),(7,2,1)\big) \\
\big((7,2,1),(1)\big) \\
*\big((1),(5,4,1)\big) \\
*\big((3),(5,2,1)\big) \\
\big((3,2,1),(5)\big) \\
\big((1,1,1),(5,2,1)\big) \\
\big((1),(5,2,1,1,1)\big) \\
\big((5),(3,2,1)\big) \\
\big((5,2,1),(3)\big) \\
\big((5,4,1),(1)\big) \\
\big((5,2,1),(1,1,1)\big) \\
\big((5,2,1,1,1),(1)\big) \\
*\big((3,2),(3,2,1)\big) \\
\big((3,2,1),(3,2)\big)
\end{array}
$$

$$
\Delta^{(2,1)}(q) = \begin{pmatrix}
q^2 & 0 & q^2 & q^3+q & q^2 & q & 0 & q^2 & 0 & 0 & 0 & 0 & q^2 & q \\
q^3 & 0 & q^3 & q^2 & 0 & q^2 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^3 & q^2 & q^3 & q^4+q^2 & q^3+q & q^2 & 0 & q^3 & q^2 & q & q & 0 & q^3 & q^2 \\
q^4 & q^3 & q^4 & q^3 & 2q^2 & q^3 & q^2 & q^2 & q^3 & q^2 & q^2 & q & 0 & q \\
0 & 0 & q^2 & q^3 & q^2 & q & 0 & 0 & 0 & q^2 & 0 & 0 & q^4 & q^3 \\
0 & 0 & q^3 & 0 & 0 & q^2 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & 0 & q^4 & q^3 & q^2 & q^3+q & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^3 & 0 & 0 & q^4 & q^3 & q^2 & 0 & q^3 & 0 & 0 & 0 & 0 & 0 & q^2 \\
q^4 & 0 & 0 & 0 & 0 & q^3 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^3 & q^4 & q^3 & q^2 & 0 & q^3 & q^2 & q^3 & q & 0 & q^5 & q^4 \\
q^3 & q^2 & 0 & q^4 & q^3 & q^2 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
q^4 & q^3 & q^4 & q^5 & q^4+q^2 & 2q^3 & q^2 & q^4 & q^3 & q^2 & 2q^2 & q & 0 & q^3 \\
0 & 0 & q^5 & 0 & q^3 & q^4 & q^3 & 0 & q^4 & q^3+q & q^3 & q^2 & 0 & 0 \\
q^5 & q^4 & 0 & 0 & q^3 & q^4 & q^3 & 0 & 0 & 0 & q^3 & q^2 & 0 & 0
\end{pmatrix}
\begin{array}{l}
\big((3,1,1),(3,2,1)\big) \\
\big((3),(3,2,1,1,1)\big) \\
\big((3,2,1),(3,1,1)\big) \\
\big((3,2,1,1,1),(3)\big) \\
\big((2,2,1),(3,2,1)\big) \\
\big((1),(3,2,2,2,1)\big) \\
\big((1,1,1),(3,2,1,1,1)\big) \\
\big((1,1,1,1,1),(3,2,1)\big) \\
\big((1),(3,2,1,1,1,1,1)\big) \\
\big((3,2,1),(2,2,1)\big) \\
\big((3,2,1),(1,1,1,1,1)\big) \\
\big((3,2,1,1,1),(1,1,1)\big) \\
\big((3,2,2,2,1),(1)\big) \\
\big((3,2,1,1,1,1,1),(1)\big)
\end{array}
$$

$$
\Delta^{(2,2)}(q) = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
q & 0 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
q^2 & q & q & 1 & . & . & . & . & . & . & . & . & . & . \\
q^2 & 0 & q & 0 & 1 & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & q & 1 & . & . & . & . & . & . & . & . \\
q^2 & q & q & 0 & q^2 & q & 1 & . & . & . & . & . & . & . \\
q^3 & q^2 & q^2 & q & 0 & 0 & q & 1 & . & . & . & . & . & . \\
0 & q & 0 & 0 & 0 & q & q^2 & q & 1 & . & . & . & . & . \\
q^3 & 0 & q^2 & 0 & q & 0 & 0 & 0 & 0 & 1 & . & . & . & . \\
q^3 & q^2 & q^2 & 0 & 0 & 0 & q & 0 & 0 & 0 & 1 & . & . & . \\
q^4 & q^3 & q^3+q & q^2 & q^2 & q & q^2 & q & 0 & q & q & 1 & . & . \\
0 & 0 & q^2 & 0 & q^3 & q^2 & 0 & 0 & 0 & q^2 & 0 & q & 1 & . \\
0 & q^2 & q^2 & q & 0 & q^2 & q^3 & q^2 & q & 0 & q^2 & q & 0 & 1
\end{pmatrix}
\begin{array}{l}
\big((3,1,1),(3,2,1)\big) \\
\big((3),(3,2,1,1,1)\big) \\
\big((3,2,1),(3,1,1)\big) \\
\big((3,2,1,1,1),(3)\big) \\
\big((2,2,1),(3,2,1)\big) \\
\big((1),(3,2,2,2,1)\big) \\
\big((1,1,1),(3,2,1,1,1)\big) \\
\big((1,1,1,1,1),(3,2,1)\big) \\
\big((1),(3,2,1,1,1,1,1)\big) \\
\big((3,2,1),(2,2,1)\big) \\
\big((3,2,1),(1,1,1,1,1)\big) \\
\big((3,2,1,1,1),(1,1,1)\big) \\
\big((3,2,2,2,1),(1)\big) \\
\big((3,2,1,1,1,1,1),(1)\big)
\end{array}
$$

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Xavier YVONNE, Institut Camille Jordan (Mathématiques), Université Lyon I, 43 Bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France.

E-mail address: `yvonne@math.univ-lyon1.fr`